Optimal Diversity Allocation in Multiuser Communication Systems—Part II: Optimization

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***Abstract*—In Part I of this paper, a class of multicarrier systems was proposed to study the effect of the method of diversity allocation on the performance of coherent multiuser communication systems operating over fading channels. In this part of the paper, optimization over the proposed class of systems is considered for a fixed number of users per unit bandwidth. The first case studied is a system where the only noise not attributable to users in the system is additive white Gaussian noise. It is observed that either a system employing exclusive allocation, where users are allocated time-bandwidth resources that are not simultaneously shared with other users, or a system employing maximum resource sharing, where all users simultaneously share time-bandwidth resources, is optimal. Next, the preferable of these two extreme forms of resource allocation is determined. For any reasonable signal-to-noise ratio (SNR) and user density, it is shown that the system employing exclusive resource allocation is optimal in a single-cell environment with perfect subchannel separation at the receiver. Finally, the optimization is repeated in the presence of partial-band interference (PBI). Once again, either a system employing exclusive resource allocation or a system employing a maximum resource sharing scheme is observed to be optimal. The presence of the PBI increases the range of user densities and SNR's where a system employing a maximum resource sharing scheme is optimal, particularly when the probability of a particular time-bandwidth slot experiencing interference is high.**

***Index Terms*—DS-CDMA, FH-CDMA, multicarrier CDMA, multipath fading channels, multiuser communications.**

I. Introduction

THIS two-part paper is motivated by the desire to find a single framework that encompasses a number of multiuser wireless communication system architectures; such a framework would conceivably allow an equitable comparison of the included architectures.

In Part I [1] of this paper, a class of multiuser multicarrier (MC) systems was developed that captures the effect of resource allocation on the performance of multiuser systems operating over fading channels. The utility of the class was demonstrated by showing that there are systems in the class which perform equivalently to a number of popular multiuser communication system alternatives. Here, optimization is performed over the class of systems when there are a fixed number of users to be supported by the system. This is equivalent to deciding how many independent time-bandwidth slots (and thus how much diversity) to allocate each user. If the number is small, users can employ time-bandwidth slots without interference from other users. However, since the total number of time-bandwidth slots in the system is fixed, the users must necessarily share time-bandwidth slots as the order of diversity allocated to each user increases.

The optimization is performed first for systems in which the only interference not attributable to system users is additive white Gaussian noise (AWGN). After completing this optimization, the optimization is reconsidered but with a chance that each time-bandwidth slot experiences additional interference, which will be denoted partial-band interference (PBI) or jamming. Under the latter conditions, it has been demonstrated in [2] that MC direct-sequence code-division multiple access (MC/DS/CDMA) is superior to standard DS/CDMA if DS/CDMA employs only the standard rake receiver. If both systems are viewed in the context of the MC framework considered here, the comparison in [2] is between two identical systems, except that the MC/DS/CDMA system employs optimal combining factors at the receiver, while the DS/CDMA system employs suboptimal combining factors. When both systems employ optimal combining, they fall in the MC class proposed here; by optimizing over this MC class, the optimization is generalized to include not only systems employing resource sharing, such as DS/CDMA and MC/DS/CDMA, but also systems employing exclusive allocation schemes.

The organization of Part II of this paper is as follows. In Section II, the class of MC systems proposed in [1] is reviewed to establish appropriate terminology. In Section III, optimization over the class of systems is performed when the only interference is multiple-access interference (MAI) and AWGN. Optimization over the class of systems when there is PBI in addition to the MAI and AWGN is considered in Section IV. Finally, Section V presents the conclusions.

II. System Definition

In this section, the channel assumptions and the proposed class of systems from Part I [1] are briefly reviewed. Readers are directed to [1] for the formal motivation, development, and interpretation of the class of systems.

Paper approved by C. Robertson, the Editor for Spread Spectrum Systems of the IEEE Communications Society. Manuscript received February 3, 1997; revised August 10, 1998 and April 29, 1999. This work was supported in part by a Rackham Predoctoral Fellowship, the National Science Foundation through a Graduate Fellowship and Grant NCR-9115869, and the Army Research Office under Contract DAAH04-95-I-0246. This work was presented in part at the 1996 Tactical Communications Conference, Fort Wayne, IN, May 1996, at the 1997 IEEE Vehicular Technology Conference, Phoenix, AZ, May 1997, and at the 1997 IEEE Military Communications Conference, Monterey, CA, November, 1997.

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Publisher Item Identifier S 0090-6778(00)00494-3.

A. Proposed Class of MC Systems

For each user, a zero-mean frequency-selective, time-nonselective fading channel is assumed that obeys the Gaussian wide-sense stationary uncorrelated scattering model presented in [3]. Because the reverse link is considered, the fading processes affecting the signals of different users will be assumed to be independent.

An MC system forms a number of narrow-band subchannels, each of which is approximately faded nonselectively; thus, it breaks the time-frequency plane available into a number of time-bandwidth slots. Denote the bandwidth of each of these slots by B and the duration of each of these slots by T_s. The methods of interleaving, coding, and user distribution across the time-bandwidth slots will define the class of MC systems presented in [1]. It is assumed that the slots employed by each user are independently faded (i.e., perfect interleaving); thus, the bandwidth of the system and frequency coherence of the channel fix a maximum diversity N available to each user. To allow for simple user separation at the receiver when time-bandwidth slots are shared, it will be assumed that each user employs repetition coding of rate $1/L$ followed by randomly generated but known binary scrambling. In other words, $b_k^i \in \{-1, +1\}$, corresponding to the ith data bit of user k, will be replicated L times and the lth replica multiplied by $a_{k,l}$, a binary random variable that is equally likely to be +1 or –1. Each of the L resulting symbols is sent over an independently faded subcarrier using power P_c, symbol duration T_s, and pulse shape $p(t)$; thus, each user achieves Lth-order diversity to mitigate the multipath fading.

The L independent slots a given user occupies will be denoted a group. To allow CDMA systems in the class, each group will be allowed to carry $K \geq 1$ users, *each signaling over all L slots in the group*. Thus, the proposed class of MC systems consists of all systems described above, such that $L \in \{1, \cdots, N\}$ and $K \in \{1, \cdots, \infty\}$. The total number of users per N slots is given as the product of the number of users per group and the number of groups as $K(N/L) = \lambda N$, where the user density is defined as $\lambda \triangleq K/L$ users/slot. The total number of users in the system is proportional to the user density λ, and it is convenient to generally characterize systems by the user density λ and the diversity L employed by each user as displayed in [1, Fig. 4].

B. Performance Characterization

Optimizations for both synchronous and asynchronous systems will be performed. In the synchronous system, the data bits of all users start at the same time; in the asynchronous system, the start time of each bit of the kth user in the group will be offset from user 0 by $u_k T_s$, where u_k is a random variable that is assumed to be uniformly distributed between 0 and 1.

Per [1], coherent demodulation with perfect phase estimation is performed on each subcarrier. The conventional receiver, which is defined as implementing the optimal decision on the demodulator outputs from the time-bandwidth slots of a given user without knowledge of the other users' data bits, demodulation outputs, timing, or spreading waveforms, is shown in [1] to exhibit bit-error probability

$$P_e^s(\lambda L, L, \Gamma) = \int_0^\infty Q\left(\sqrt{2\overline{\gamma}_s(\lambda L,\, L,\, \Gamma) f}\right) \frac{f^{L-1}\, e^{-f}}{(L-1)!}\, df \tag{1}$$

where $Q(x) = \int_x^\infty 1/\sqrt{2\pi}\, e^{(-y^2)/2}\, dy$, and the signal-to-interference ratio (SIR) per subchannel $\overline{\gamma}_s(\lambda L,\, L,\, \Gamma)$ is given by

$$\begin{aligned}\overline{\gamma}_s(\lambda L,\, L,\, \Gamma) &= \frac{\Gamma}{\Gamma(\lambda L - 1) + L}\\ &= \begin{cases} \dfrac{1}{\zeta_s L - 1}, & L \in \left[\dfrac{1}{\lambda}, \infty\right) \\ \dfrac{\Gamma}{L}, & L \in \left[1, \dfrac{1}{\lambda}\right). \end{cases}\end{aligned} \tag{2}$$

The received signal-to-noise ratio (SNR) is given by $\Gamma \triangleq E_s/N_0 E[\alpha_{0,\,l}^2]$, where $E_s = LP_cT_s$ is the transmitted energy per bit and $\zeta_s \triangleq \lambda + 1/\Gamma$. There are two cases above because of the restriction $\lambda L - 1 \geq 0$. The integral on the right-hand side of (1) can be evaluated in the same manner as the single-user case [44, p. 723] to yield

$$P_e^s(\lambda L,\, L,\, \Gamma) = \left(\frac{1-v}{2}\right)^L \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left(\frac{1+v}{2}\right)^k \tag{3}$$

where for a coherent binary phase-shift keying system

$$v = \sqrt{\frac{\overline{\gamma}_s(\lambda L,\, L,\, \Gamma)}{1 + \overline{\gamma}_s(\lambda L,\, L,\, \Gamma)}}. \tag{4}$$

The finite series of positive terms in (3) is well suited to numerical evaluation, and all plots displayed in this paper will be based on this equation or its equivalent for the asynchronous system. However, for analytical purposes, (1) is more useful as demonstrated in [5]. The goal is to minimize (1) for a fixed λ and Γ, which will be simplified by the fact that (1) depends on these parameters only through the parameter ζ_s when $\lambda L > 1$.

If it is assumed that the MAI is Gaussian when conditioned on the fading of the desired user, an approximation to the bit-error probability of the conventional receiver in the asynchronous system is obtained as [1]

$$\tilde{P}_e^a(\lambda L, L, \Gamma) = \int_0^\infty Q\left(\sqrt{2\overline{\gamma}_a(\lambda L, L, \Gamma) f}\right) \frac{f^{L-1}}{(L-1)!}\, e^{-f}\, df \tag{5}$$

where

$$\begin{aligned}\overline{\gamma}_a(\lambda L,\, L,\, \Gamma) &= \frac{\Gamma}{\psi\Gamma(\lambda L - 1) + L}\\ &= \begin{cases} \dfrac{1}{\zeta_a L - \psi}, & L \in \left[\dfrac{1}{\lambda}, \infty\right) \\ \dfrac{\Gamma}{L}, & L \in \left[1, \dfrac{1}{\lambda}\right) \end{cases}\end{aligned} \tag{6}$$

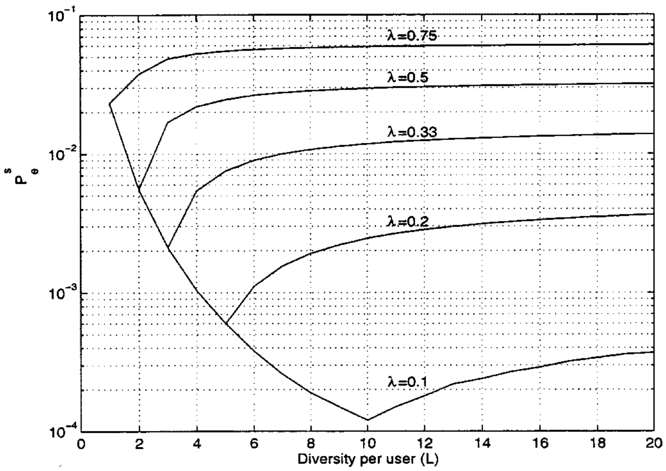

Fig. 1. Bit-error probability of the conventional receiver in a synchronous system for diversity L and user density $\lambda < 1$ users/slot at received SNR $\Gamma = 10$ dB.

where $\zeta_a = \psi\lambda + 1/\Gamma$, and ψ depends on the shape of the signaling waveform through

$$\psi = E_{u_k}\left[\left(\int_0^{u_k} \hat{p}(1-\tau+s)\hat{p}(s)\,ds\right)^2 + \left(\int_{u_k}^{1} \hat{p}(s-u_k)\hat{p}(s)\,ds\right)^2\right]. \quad (7)$$

where $\hat{p}(x) = p(xT_s)$.

An explicit relation between the approximation to the bit-error probability of the conventional receiver in the asynchronous system and the bit-error probability of the conventional receiver of the synchronous system is given by

$$\tilde{P}_e^a(\lambda L, L, \Gamma) = P_e^s(\psi\lambda L + 1 - \psi, L, \Gamma). \quad (8)$$

It will be demonstrated that the approximation derived in this section will serve as a useful crutch to derive optimization results.

III. Optimization for System with AWGN Interfernce

A. Optimization—Synchronous Systems

Assume that the user density λ is fixed and consider the choice of L, which corresponds to the method of time-bandwidth slot allocation that yields the minimal bit-error probability.

1) Small User Densities ($\lambda < 1$): Fig. 1 contains plots of the bit-error probability of the conventional receiver in a synchronous system for values of $\lambda < 1$. It appears that the optimal system employs $L = 1/\lambda$ (or $K = 1$). It is clear that systems with smaller values of L than $1/\lambda$ have smaller diversity than a system employing $L = 1/\lambda$ at the same SNR Γ; thus, the $L = 1/\lambda$ system is superior to these for any Γ. The rest of this section supports the notion that systems with higher values of L are suboptimal for any reasonable Γ.

The problem may be explored analytically by replacing the $Q(\cdot)$ function in (1) with an upper bound and optimizing the resulting upper bound to the error probability. The bound employed is given by $Q(\beta) \le 1/2\, e^{(-\beta^2)/2}$ [6, p. 123]. The bit-error probability obtained when this bound is employed is given by

$$P_{ch}^s(\lambda L, L, \Gamma) = \frac{1}{2(1+\overline{\gamma}_s(\lambda L, L, \Gamma))^L}. \quad (9)$$

Although this bound is approximately an order of magnitude loose, it preserves the relative difference between the system bit-error probabilities for all cases studied [5]. It is then straightforward to establish that this bound is minimized at $L = 1/\lambda$ for all $\Gamma > 0$ and $\lambda > 0$ through the following steps. $P_{ch}^s(\lambda L, L, \Gamma)$ is minimized when $(1 + 1/(\zeta_s L - 1))^L$ is maximized. Consider L as a continuous variable and let

$$g(L) \triangleq \ln\left(1 + \frac{1}{\zeta_s L - 1}\right)^L = L \ln\left(1 + \frac{1}{\zeta_s L - 1}\right). \quad (10)$$

Then

$$\frac{dg(L)}{dL} = \frac{-1}{\zeta_s L - 1} + \ln\left(1 + \frac{1}{\zeta_s L - 1}\right) \quad (11)$$

$$\le \frac{-1}{\zeta_s L - 1} + \frac{1}{\zeta_s L - 1} \quad (12)$$

$$= 0 \quad (13)$$

where the inequality $\ln x \le x - 1$ has been used in the second line. Thus, $g(L)$ is decreasing in L, implying $P_{ch}^s(\lambda L, L, \Gamma)$ is minimized at $L = 1/\lambda$. Note that the same mathematical argument can be used to demonstrate that the soft-decision Bhattacharyya parameter of the system is minimized when $L = 1/\lambda$ [7].

2) Large User Densities ($\lambda > 1$): For the case $\lambda \ge 1$, $L \in [1, N]$, the bit-error probability of the conventional receiver is given succinctly by

$$P_e^s(\lambda L, L, \Gamma) = \int_0^\infty Q\left(\sqrt{\frac{2f}{\zeta_s L - 1}}\right) \frac{f^{L-1}\, e^{-f}}{(L-1)!}\, df. \quad (14)$$

Thus, the system conditions are represented by the single parameter ζ_s. It is possible to study the optimal values of L for extreme values of ζ_s. First, for $\zeta_s = 1$ (i.e., $\lambda = 1$ and $\Gamma = \infty$), zero error probability can be obtained for $L = 1$, thus leading to the notion that small ζ_s will probably require small L. For large ζ_s, $\zeta_s L - 1 \approx \zeta_s L$, which transforms (14) into the bit-error probability of the conventional receiver in a single-user system operating over L Rayleigh nonselectively-faded slots with SNR $1/\zeta_s$, in which case the bit-error probability is minimized at $L = \infty$.

For moderate values of ζ_s, it is difficult to perform the optimization directly. However, Fig. 2 suggests that for N large, $L = 1$ or $L = N$ is optimal. Furthermore, the $L = N$ (for $N > 10$) case is well approximated by the $L = \infty$ case. Thus, in this section, $P_e^s(\lambda, 1, \Gamma)$ and $\lim_{L\to\infty} P_e^s(\lambda L, L, \Gamma)$ are compared. Trivially

$$P_e^s(\lambda, 1, \Gamma) = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\Gamma}{\Gamma\lambda + 1}} = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{1}{\zeta_s}} \quad (15)$$

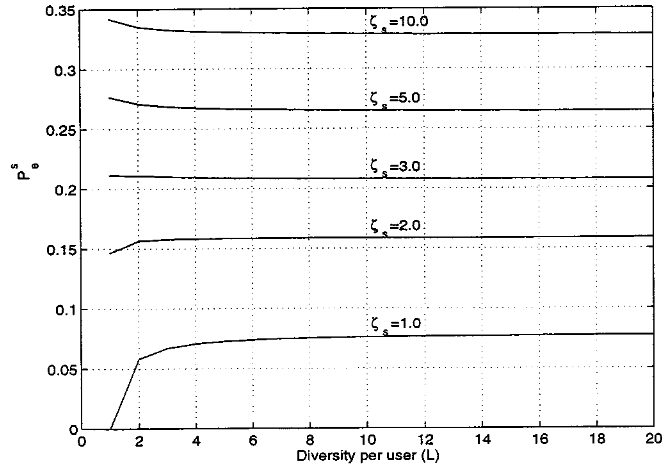

Fig. 2. Bit-error probability of the conventional receiver in the synchronous system employing diversity L with channel parameter ζ_s for $\lambda > 1$.

and [1]

$$\lim_{L\to\infty} P_e^s(\lambda L, L, \Gamma) = Q\left(\sqrt{\frac{2\Gamma}{\Gamma\lambda+1}}\right) = Q\left(\sqrt{\frac{2}{\zeta_s}}\right). \quad (16)$$

Now, the comparison is straightforward. $L = 1$ is preferable whenever

$$\frac{1}{2} - \frac{1}{2}\sqrt{\frac{1}{\zeta_s}} \leq Q\left(\sqrt{\frac{2}{\zeta_s}}\right). \quad (17)$$

Unfortunately, this is not an easy equation to solve. However, the convexity of $Q(p)$ guarantees that it intersects a straight line at only two points. One of these is at $p = 0$, so there is only one other intersection (denote it ζ_s^0) to find. It is straightforward to observe that the $L = 1$ system is optimal up to ζ_s^0 and the $L = \infty$ system afterward. Employing an interesting property of the series representation of the $Q(\cdot)$ function, this point can be analytically bounded as $2.59 \leq \zeta_s^0 \leq 2.63$ [5], which agrees with Fig. 2. Thus, it requires a very large value of ζ_s for preference to be given to the maximum shared allocation system over the exclusive allocation scheme. Finally, note from Fig. 3 a supporting result. For a user density of $\lambda = 1.0$, the diversity increase afforded by the maximum resource sharing system is more than offset by the error floor introduced by the MAI. However, this result should be taken with a bit of caution, as the results are highly sensitive to the system assumptions. The results change drastically in an asynchronous system per the next section or in a system employing multiuser receivers [5].

B. Optimization—Asynchronous Systems

If the conditional Gaussian approximation is employed as given by (5), similar results to the synchronous case are obtained in that the exclusive allocation ($L = 1$) system or maximum shared allocation ($L = N$) system is optimal for all Γ, λ, and reasonable ψ. Note that the conditional Gaussian approximation is exact for exclusive allocation systems ($L = 1/\lambda$, $\lambda \leq 1$) and asymptotic maximum resource sharing systems ($L = \infty$) [1], [5]. Thus, whenever the approximation is minimized at $L = 1/\lambda$ for $\lambda \leq 1$, or at $L = \infty$ for any λ, it yields the true error probability of the system at its minimum, which is the true minimum

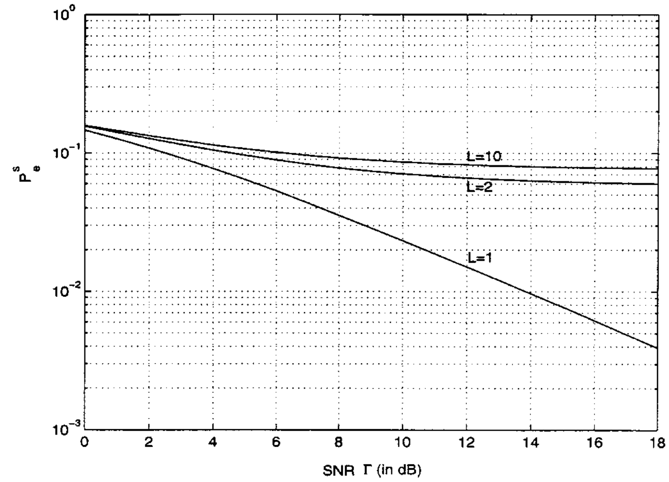

Fig. 3. Bit-error probability of the conventional receiver in the synchronous system with user density $\lambda = 1.0$ for various diversity per user L. Note that the exclusive allocation system is optimal across all SNR, as the diversity increase obtained by the maximum resource sharing system is offset by the error floor introduced by the MAI.

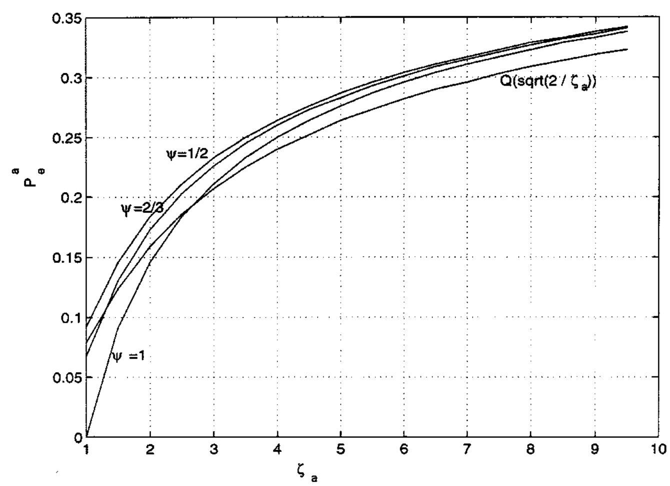

Fig. 4. Approximation to the bit-error probability of the conventional receiver in the asynchronous system employing diversity $L = 1$, pulse shape parameter ψ, and channel parameter ζ_a. Recall that the parameter ζ_a depends on ψ, and thus the curves for various ψ should not be compared to one another to compare relative performance.

under the tenet that the conditional Gaussian approximation is optimistic [8], [9].

Finding the cutoff point ζ_a^0, where the preferred system changes from exclusive allocation to maximum shared allocation, is more difficult to perform analytically than in the synchronous case unless $\psi = 1$. It involves the comparison of

$$P_e^a(\lambda L, L, \Gamma) = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{1}{\zeta_a + (1-\psi)}} \quad (18)$$

and

$$\lim_{L\to\infty} P_e^a(\lambda L, L, \Gamma) = Q\left(\sqrt{\frac{2}{\zeta_a}}\right). \quad (19)$$

For a fixed ζ_a, the latter function is fixed for all ψ, whereas the first depends on ψ. Fig. 4 performs the comparison. Note that as ψ increases (more multiuser interference for shared schemes), the exclusive scheme becomes preferable for larger regions of ζ_s. Fig. 4 can be used in a more general context, including the effects of intercell interference and voice activity in a cellular system. The interested reader is referred to [5, ch. 4].

IV. Optimization in PBI

As demonstrated in [1], the MC/DS/CDMA systems of [2] fit under the MC framework studied here. In [2], the emphasis is on the comparison of DS/CDMA and MC/DS/CDMA, and it is shown that MC/DS/CDMA shows markedly better performance in PBI. This is directly attributable to the fact that the DS/CDMA system is not allowed to perform interference rejection despite having knowledge at the receiver of the bands experiencing PBI. Thus, the relation of the results is as expected. In this work, a different approach is employed to optimization in partial-band jamming. Each system in the MC framework is allowed to use optimal combining at the receiver; thus, the question posed is what type of resource allocation is inherently preferable in partial-band jamming.

The model will be precisely the same as in the AWGN case except that it will be assumed that any given slot experiences PBI with probability ρ. This PBI will be modeled as white noise with two-sided power spectral density $N_J/2\rho$, thus keeping the average power constant over ρ. Although it is not necessary to keep the average power constant over ρ for the work considered in this paper, it will allow consideration of worst case PBI with minimal additional effort.

Given the above assumption, the L slots employed by a given user can be of one of two types—jammed or not jammed. Maximum-ratio combining with knowledge of the fading and jamming status of each slot is performed; in other words, the correlator output of a given subchannel is multiplied by the magnitude of the fading on the subchannel divided by the variance of the total interference on the subchannel, and the results are summed to form the decision statistic. Conditioning on the number of slots j jammed of a given user, the probability of error of the synchronous system can be written as

$$P_{e,\,j}^{S}(\lambda L,\, L,\, \Gamma) = E_{G_j}\left[Q\,\sqrt{2G_j}\right] \tag{20}$$

where G_j [4, p. 734] is the sum of L independent exponential random variables, $L-j$ of which have density

$$p_N(x) = \frac{1}{\overline{\gamma}_s(\lambda L,\, L,\, \Gamma)}\, \exp\left(\frac{-x}{\overline{\gamma}_s(\lambda L,\, L,\, \Gamma)}\right) \tag{21}$$

and j of which have density

$$p_J(x) = \frac{1}{\overline{\gamma}_s^J(\lambda L,\, L,\, \Gamma)}\, \exp\left(\frac{-x}{\overline{\gamma}_s^J(\lambda L,\, L,\, \Gamma)}\right) \tag{22}$$

where

$$\overline{\gamma}_s^J(\lambda L,\, L,\, \Gamma) = \frac{1}{\left(\lambda + \frac{1}{\Gamma} + \frac{N_J}{\rho E_s E[\alpha_{k,\,l}^2]}\right) L - 1}. \tag{23}$$

TABLE I

Point ζ_s^0 Below Which the Exclusive Allocation Scheme is Optimal and Above Which the Maximum Sharing Allocation Scheme is Optimal for Jammer to-Signal Ratio $N_J/E_sE[\alpha_{k,\,l}^2]$ and Slot Jamming Probability ρ

$\frac{N_J}{E_sE[\alpha_{k,l}^2]}$	ρ	$\zeta_s^0 \pm 0.05$
0 dB	0.05	2.30
	0.1	2.20
	0.2	1.95
	0.5	1.70
10 dB	0.05	2.15
	0.1	1.75
	0.2	1.30
	0.5	0.25
20 dB	0.05	2.10
	0.1	1.70
	0.2	1.10
	0.5	0.15

To our knowledge, the probability of error in (20) has not been previously evaluated in closed form, although a similar expression is required in [2]. This is because it does not fall into the two cases of maximum-ratio combining of Rayleigh-faded slots for which the closed-form solutions are well known: all slots with the same SNR [4, p. 723] and each slot with a distinct SNR [4, p. 734] . The closed-form expression for (20) is derived in the appendix as

$$\begin{aligned} P_{e,\,j}^{S}&(\lambda(i+j),\, i+j,\, \Gamma)\\ &= \sum_{l=0}^{i-1} \binom{l+j-1}{l} \frac{(-1)^l}{\left(1-\frac{\mu_b^2}{\mu_a^2}\right)^j}\\ &\quad\cdot\left[\frac{1}{\left(\frac{\mu_a^2}{\mu_b^2}-1\right)^l} U(i-l,\, v_a)\right.\\ &\quad - \sum_{m=0}^{l+j-1}\binom{m+i-1-l}{m}\left(\frac{\mu_b^2}{\mu_a^2}\right)^i\\ &\quad\left.\cdot\left(1-\frac{\mu_b^2}{\mu_a^2}\right)^{m-l} U(m+i-l,\, v_b)\right] \end{aligned} \tag{24}$$

where

$$U(k,\, v) = \left(\frac{1-v}{2}\right)^k \sum_{p=0}^{k-1}\binom{k-1+p}{p}\left(\frac{1+v}{2}\right)^p \tag{25}$$

$$v_a = \sqrt{\frac{\mu_a^2}{1+\mu_a^2}} \qquad v_b = \sqrt{\frac{\mu_b^2}{1+\mu_b^2}} \tag{26}$$

and the average SNR's per subchannel are given by $\mu_a^2 = \overline{\gamma}_s(\lambda L,\, L,\, \Gamma)$ and $\mu_b^2 = \overline{\gamma}_s^J(\lambda L,\, L,\, \Gamma)$.

Although the resulting expression is complicated, its form as a finite sum of elementary functions is quite amenable to numerical evaluation, and should prove useful for partial-band jamming analyzes in general. The unconditional probability of error is given by $P_e^{S,J}(\lambda L, L, \Gamma) = \sum_{j=0}^{L} \binom{L}{j} \rho^j (1-\rho)^{L-j} P_{e,j}^S(\lambda L, L, \Gamma)$. The same observation is made from the numerical results as made in the optimization in the presence of AWGN of Section III—the exclusive allocation ($L = 1/\lambda$) or the maximum shared allocation ($L = N$) is optimal. In Table I, the point ζ_s^0 below which the exclusive allocation scheme is optimal and above which the maximum shared allocation scheme is optimal is shown. In general, the more severe the jamming, the more favorable is the maximum shared allocation scheme. Note, however, that the shift is not large unless the probability of a given slot being hit is high. The reason for this is as follows: for small λ, the exclusive scheme has a number of slots ($1/\lambda$) for use. If one is jammed, that slot is weighted only slightly and the others are able to accomplish communication. For larger λ, the penalty paid when the single slot employed by the exclusive allocation scheme is jammed is not as steep, because the error rates are high enough that the ratio of the probability of error of a single jammed slot to a single unjammed slot is not large. However, if the probability of jamming of each slot is somewhat large, the probability of error when all slots are jammed dominates the error probability, which favors the maximum sharing scheme per the results from the AWGN case.

V. Conclusions

In Part II of this paper, the proposed class of systems presented in Part I [1] to study the optimal allocation of time-bandwidth slots to users operating over multipath fading channels was used as an optimization framework to minimize the bit-error probability of the conventional receiver when a fixed number of users are in the system. This was done for two separate cases. For the system experiencing only AWGN interference, it was observed that either the exclusive allocation scheme or the maximum sharing scheme is optimal. The point at which the optimal system shifts from the exclusive allocation scheme to the maximum shared allocation scheme for a synchronous system (or an asynchronous system with $\psi = 1$) suggests that the exclusive allocation scheme is optimal for any reasonable communication system. However, for asynchronous systems, the variance of the multiuser interference is scaled by an additional factor ψ, which can produce dramatic changes in the results.

The second environment studied included PBI. For this case, a complicated but closed-form expression was derived that allows for fast numerical calculation of the system error probability. It was observed that a system employing either the exclusive allocation scheme or the maximum resource sharing scheme is optimal, with a shift down from the AWGN case of the point below which the exclusive allocation scheme is optimal. The shift is proportional to the strength of the interference; however, it is less than may be expected because the maximum-ratio combiner effectively notch filters out the jammed subchannels at small user densities, where the penalty of a jammed subchannel is high. For a fixed jammer power, the maximum shared resource system becomes more favorable as the jammer spreads his power out across many bands.

Appendix
Probability of Error in Partial Band Jamming

In this appendix, the probability of error is derived for a binary coherent system operating over $i+j$ independent Rayleigh-faded slots, where j of the slots have average SIR μ_b^2 , and the remaining i slots have average SIR μ_a^2, where $\mu_a^2 > \mu_b^2$. The decision statistic of the maximal-ratio combiner is the weighted sum of the outputs of the slots, where the weighting of each slot is by the slot fading magnitude divided by the variance of the interference on the slot. From (20)–(22), the SNR of the maximum-ratio combiner is given by $Z = X + Y$, where X is a central chi-square random variable with $2i$ degrees of freedom that is obtained from the summation of the squares of $2i$ zero-mean independent Gaussian random variables with variance $(1/2)\mu_a^2$, and Y is a central chi-square random variable with $2j$ degrees of freedom that is obtained from the summation of the squares of $2j$ zero-mean independent Gaussian random variables with variance $(1/2)\mu_b^2$. Thus

$$p_X(x) = \frac{1}{(\mu_a^2)^i (i-1)!} \, x^{i-1} \, e^{-x/\mu_a^2}$$

and

$$p_Y(y) = \frac{1}{(\mu_b^2)^j (j-1)!} \, y^{j-1} \, e^{-y/\mu_b^2}.$$

If $j = 0$ or $i = 0$, then $p_Z(z) = p_X(z)$ or $p_Z(z) = p_Y(z)$, respectively. If $j > 1$ and $i > 1$, the situation is much more complicated. The density of the sum of two independent random variables is given as the convolution of the component densities. Recognizing that the random variables are positive yields (for $z > 0$)

$$\begin{aligned} p_Z(z) &= \int_0^z p_X(z-x) p_Y(x) \, dx \\ &= \frac{e^{-z/\mu_a^2}}{(\mu_b^2)^j (\mu_a^2)^i (i-1)!(j-1)!} \int_0^z (z-x)^{i-1} \\ &\quad \cdot \exp\left[-x\left(\frac{1}{\mu_b^2} - \frac{1}{\mu_a^2}\right)\right] x^{j-1} \, dx. \end{aligned} \tag{27}$$

The above integral can be evaluated to yield (employ [10, eq. (8.384.6)] in the result of [10, eq. (3.383.1)]

$$\begin{aligned} p_Z(z) = &\frac{e^{-z/\mu_a^2}}{(\mu_b^2)^j (\mu_a^2)^i (i+j-1)!} \, z^{i+j-1} \\ &\cdot F\left(j, \, i+j; \, \left(\frac{1}{\mu_b^2} - \frac{1}{\mu_a^2}\right) z\right) \end{aligned} \tag{28}$$

where $F(\cdot, \cdot; \cdot)$ is the degenerate hypergeometric function, which is defined as [10, eq. (9.210.1)]

$$\begin{aligned} F(\alpha, \beta; u) \triangleq 1 &+ \frac{\alpha}{\beta} \frac{u}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} \frac{u^2}{2!} \\ &+ \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} \frac{u^3}{3!} + \cdots \end{aligned}$$

Unfortunately, due to the presence of the degenerate hypergeometric function, $p_Z(z)$ is not in closed form, although the above expression is useful when $(1/\mu_b^2 - 1/\mu_a^2)z$ is small. Although the first two arguments of the degenerate hypergeometric function are integers, we are unable to manipulate it directly into a finite series. Thus, return to (27) and take a slightly different approach. If a binomial expansion of $(z-x)^{i-1}$ is employed, the integral moved through the resulting summation, and integration performed using [10, eq. (3.381.1)] in conjunction with [10, eq. (8.352.1)] yields

$$p_Z(z) = \frac{e^{-z/\mu_a^2}}{(\mu_b^2)^j(\mu_a^2)^i(i-1)!(j-1)!}\sum_{l=0}^{i-1}\binom{i-1}{l}(-1)^l z^{i-1-l} \cdot \frac{1}{\left(\frac{1}{\mu_b^2}-\frac{1}{\mu_a^2}\right)^{l+j}}(l+j-1)! \cdot \left(1-\exp\left[-z\left(\frac{1}{\mu_b^2}-\frac{1}{\mu_a^2}\right)\right] \cdot \sum_{m=0}^{l+j-1}\frac{z^m\left(\frac{1}{\mu_b^2}-\frac{1}{\mu_a^2}\right)^m}{m!}\right)$$

which will be more useful if the powers of z are grouped together explicitly. Doing this and simplifying slightly yields

$$p_Z(z) = \sum_{l=0}^{i-1}\binom{l+j-1}{l}\frac{(-1)^l}{(i-1-1)!} \cdot \frac{1}{(\mu_a^2)^i\left(\frac{1}{\mu_b^2}-\frac{1}{\mu_a^2}\right)^l\left(1-\frac{\mu_b^2}{\mu_a^2}\right)^j} \cdot \left(z^{i-1-l}e^{-z/\mu_a^2} - \sum_{m=0}^{l+j-1} \cdot \frac{z^{m+i-1-l}e^{-z/\mu_b^2}\left(\frac{1}{\mu_b^2}-\frac{1}{\mu_a^2}\right)^m}{m!}\right).$$

Although the resulting expressing is quite intimidating, it has the nice property that it consists only of finite sums and elementary functions, making numerical evaluation simple. More importantly, this function can be integrated over the $Q(\cdot)$ function to find $P_{e,j}^s(\lambda(i+j), i+j, \Gamma) = \int_0^\infty Q(\sqrt{2z})p_Z(z)\,dz$ as [4, p. 723] ‘

$$P_{e,j}^S(\lambda(i+j), i+j, \Gamma) = \sum_{l=0}^{i-1}\binom{l+j-1}{l}\frac{(-1)^l}{\left(1-\frac{\mu_b^2}{\mu_a^2}\right)^j} \cdot \left[\frac{1}{\left(\frac{\mu_a^2}{\mu_b^2}-1\right)^l}U(i-l, v_a) - \sum_{m=0}^{l+j-1} \cdot \binom{m+i-1-l}{m}\left(\frac{\mu_b^2}{\mu_a^2}\right)^i \cdot \left(1-\frac{\mu_b^2}{\mu_a^2}\right)^{m-l}U(m+i-l, v_b)\right]$$

where

$$U(k, v) = \left(\frac{1-v}{2}\right)^k\sum_{p=0}^{k-1}\binom{k-1+p}{p}\left(\frac{1+v}{2}\right)^p$$

$$v_a = \sqrt{\frac{\mu_a^2}{1+\mu_a^2}}$$

and

$$v_b = \sqrt{\frac{\mu_b^2}{1+\mu_b^2}}.$$

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers, whose careful consideration of the manuscript improved the presentation of the material.

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