Improved Upper Bounds on the Packet Error Probability of Slotted and Unslotted DS/SS Systems

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***Abstract*—We consider a direct-sequence spread-spectrum multiple access communication system with convolutional coding and hard decision decoding. We calculate the packet error probability and throughput of this system. Previous bounds on packet error probability have relied on the worst case assumption of phase and chip synchronous interference. We present new bounds on packet error probability and throughput for the case of both slotted and unslotted systems. These new bounding techniques are based on two advances: an improved Chernoff bound on the error event probability of a convolutional code, and the use of moment space techniques. Numerical results indicate that these new bounds on packet error probability improve on previously reported bounds by more than an order of magnitude. We also examine the problem of choosing the optimum code rate which maximizes throughput. We compare the optimum code rate which results from the bounding technique to the optimum code rate derived from an approximation technique. Although the bounding technique and approximation technique yield very different results for throughput, the resulting choice of optimum code rate is similar.**

I. Introduction

DIRECT-SEQUENCE spread-spectrum (DS/SS) communication systems have found increasing application to military and commercial communications. Much research effort has focused on the development of techniques to analyze the performance of these systems, and there are now good techniques available for computing bit error rate [4], [9]. However, when DS/SS is used in a packet radio system, a single uncorrected bit error will render an entire block of data useless. In this case, the performance measure of interest is packet error probability.

If multiple access (MA) interference is present, powerful error correction coding is required for a DS/SS MA system to achieve acceptable packet error probability. This coding takes the form of either a block code or a convolutional code. We further classify a DS/SS MA system as either *slotted* or *unslotted*, depending on whether all users transmit packets simultaneously during time *slots* of fixed duration.

In previous work, Pursley and Taipale [10] have computed a bound on packet error probability for slotted systems with convolutional coding. Storey and Tobagi [12] have extended these results to the case of unslotted systems. Joseph and Raychaudhuri [3] have examined the packet error probability of unslotted systems with block coding. The primary reason that the packet error probability calculation is not a straightforward extension of the bit error probability calculation is that bit errors within the same packet are not independent events. In [10] and [12], this problem is avoided by assuming a worst case value for bit error probability. As a result, the overall bound on packet error probability may be loose. In [3], results are computed based on the approximation that bit error events are independent.

In [5], Morrow and Lehnert follow a novel approach to evaluate the packet error probability of a slotted system with block coding. They apply the theory of moment spaces to compute a tighter bound on the packet error probability than the bound obtained with the techniques of [10], [12], [3], without ignoring the dependencies which may exist between bit error events within the same packet. This bound on packet error probability may be used either with a true bound on bit error rate or with the "improved Gaussian approximation" for bit error rate which they also develop in [5]. Morrow and Lehnert further develop these results in [6].

In this paper, we demonstrate how these techniques can be used to improve on the bounds of [10] and [12]. Unlike [5] and [6], we consider a DS/SS MA system which employs convolutional coding and a hard decision Viterbi decoder at the receiver. Using moment space techniques, we present improved bounds on packet error probability and throughput for both slotted and unslotted systems. We also present results on the selection of the optimum code rate for these systems. Throughout this paper, we assume that power control is employed so that equal signal energy is received from each transmitter.

The paper is organized as follows. In Section II, we describe the system model under consideration. In Section III, we present new, tighter bounds for a slotted system. In Section IV, we extend those results to the case of unslotted systems. We illustrate these new bounds on packet error probability and throughput in Section V. Section VI concludes this paper.

II. System Model

In this section, we outline the model used in this paper for a DS/SS packet transmission system with multiple-access interference. We also review the techniques available for computing bit error probability, and describe simple traffic models for slotted and unslotted packet radio systems.

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A. DS/SS MA System

We consider a DS/SS MA system with binary phase shift keyed (BPSK) signaling, and a correlation receiver. Our model is based on [8]. There are a total of K users transmitting over a common channel. Associated with each user $k \in \{1, \cdots, K\}$ is a data signal $b_k(t)$ and a signature waveform $a_k(t)$ which are functions of time. These are defined by

$$b_k(t) = \sum_{i=-\infty}^{\infty} b_{k,i}\psi_T(t - iT) \tag{1}$$

$$a_k(t) = \sum_{j=-\infty}^{\infty} a_{k,j}\psi_{T_c}(t - jT_c), \tag{2}$$

where $\{b_{k,i} \in \{+1, -1\}\}$ is an infinite sequence of encoded data bits, $\{a_{k,j}\}$ is an infinite random signature sequence with each chip $a_{k,j}$ independent and equiprobably distributed on $\{+1, -1\}$, and $\psi_T(\cdot)$ is the unit pulse function of duration T, defined by

$$\psi_T(t) = \begin{cases} 1, & t \in [0, T) \\ 0, & \text{else.} \end{cases} \tag{3}$$

The duration of each encoded data bit is T, while the duration of each chip in the signature signal is T_c. As a result the number of chips per bit is $N = T/T_c$, where N is a integer.

Each user generates a signal $s_k(t)$ by modulating the data signal by its signature signal and a carrier waveform, with the result

$$s_k(t) = \sqrt{2P}\cos(\omega_c t + \phi_k)a_k(t)b_k(t), \tag{4}$$

where P is the signal power, ω_c is the carrier frequency, and ϕ_k is a random phase, uniformly distributed on the interval $[0, 2\pi)$.

A correlation receiver receives the signal $r(t)$which is the sum of delayed versions of all transmitted signals and thermal noise. The received signal $r(t)$ is

$$r(t) = n(t) + \sum_{k=1}^{K} s_k(t - \tau_k) \tag{5}$$

where $n(t)$ is a white Gaussian process with two-sided power spectral density $N_o/2$, and τ_k is a random delay, uniformly distributed on $[0, T)$. This channel model is illustrated in Fig. 1. The synchronous correlation receiver recovers the transmitted data bit by correlating $r(t)$ with a local version of the transmitted signal to form a decision statistic $Z_{k,i}$ where

$$Z_{k,i} = \int_{iT+\tau_k}^{(i+1)T+\tau_k} r(t)\cos[\omega_c(t-\tau_k)+\phi_k]a_k(t-\tau_k)dt. \tag{6}$$

The decision statistic is used to form an estimate $\hat{b}_{k,i}$ of the data bit $b_{k,i}$ based on the rule

$$\hat{b}_{k,i} = \begin{cases} 1, & Z_{k,i} \geq 0 \\ -1, & Z_{k,i} < 0. \end{cases} \tag{7}$$

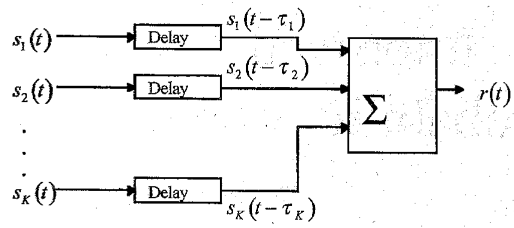

Fig. 1. Model of DS/SS MA system.

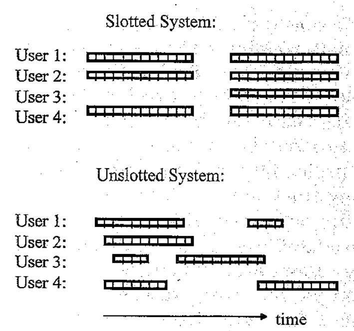

Fig. 2. Slotted and unslotted transmission.

B. Traffic Models

In a packet transmission system, blocks of data are grouped into blocks of consecutive bits called packets. We assume that one packet has a duration of L encoded bits. A single uncorrected error forces retransmission of the entire packet of data. We classify a packet transmission system as either *slotted* or *unslotted*, depending on whether all active users transmit packets simultaneously during time slots of fixed duration. Slotted and unslotted systems are illustrated in Fig. 2.

In a slotted system, time is divided into slots of duration LT which is the duration of a packet. The number of users K is fixed over the duration of the entire packet. Although the users are coarsely synchronized in that they commence and cease transmission at approximately the same time, they are not finely synchronized. The phases $\{\phi_k\}$ and delays $\{\tau_k\}$ of the interfering users are still modeled as random variables. We assume an infinite population of potential users, and the number of users K that transmit during a given slot is a Poisson random variable with probability mass function $p_K(k)$, where

$$p_K(k) = \frac{G^k \exp(-G)}{k!}, \qquad k = 0, 1, \cdots \tag{8}$$

and the expected number of users G is called the offered traffic. A packet is said to be successful if, after error correction from the convolutional code, there are no errors in the packet.

In an unslotted system, users commence and cease transmission independently, and the number of interferers can vary during the transmission of a packet. The length L of each new packet is modeled as being exponentially distributed with mean $\bar{L}$ data bits as in [12]. Since actual packets lengths must be integers, we use a discrete approximation to an exponential distribution for our numerical results, taking $\Pr[L = l] = p_L(l) = \int_{l-1}^{l} \frac{1}{\bar{L}} \exp -x/\bar{L} dx$. New users from an infinite population begin transmission according to a

Poisson process with arrival rate λ. Since packets arrive in the system according to a Poisson process and have exponentially distributed length, the distribution of active users is identical to that for an $m/m/\infty$ queue. It is shown in [1] that the number of users at any given time has the Poisson distribution of (8), with $G = \lambda \bar{L}$.

C. Performance Measures

This section discusses the performance measures of interest in this paper. It is convenient to condition the probability of bit error on the delays $\tau = \{\tau_1, \cdots, \tau_K\}$ and phases $\phi = \{\phi_1, \cdots, \phi_K\}$. We express this conditional probability as $\rho_K(\tau, \phi) = \Pr[\hat{b}_{k,i} \neq b_{k,i} \mid \tau, \phi]$. Note that the random processes $n(t)$ and $a_k(t)$ are independent from one data interval to the next. Therefore, when conditioned on the delays τ and phases ϕ, all bit error events are nearly independent of one another.[1] The average bit error probability is found by taking the expected value of $\rho_K(\tau, \phi)$ over these phases and delays

$$\bar{\rho}_K = E_{\tau,\phi}[\rho_K(\tau, \phi)]. \tag{9}$$

In [4], a technique is presented for efficiently evaluating the multidimensional integral implied by (9) to an arbitrary level of accuracy.

It is shown in [10] that when all users have equal signal power the worst bit error rate occurs when all multiple-access interferers are phase and chip synchronous ($\tau = \mathbf{0}$ and $\phi = \mathbf{0}$) with the desired signal. We denote the bound on $\rho_K(\tau, \phi)$ obtained from this fact as ρ_K^U and write

$$\rho_K(\tau, \phi) \leq \rho_K^U = \rho_K(\mathbf{0}, \mathbf{0}), \qquad \forall \tau, \phi. \tag{10}$$

If it is phase and chip synchronous, the multiple access interference has a binomial probability distribution and ρ_K^U can be computed from the formula

$$\rho_K^U = \sum_{j=0}^{(K-1)N} \binom{(K-1)N}{j} 2^{(1-K)N} q(j), \tag{11}$$

where $q(j)$ is given by

$$q(j) = Q\left(\sqrt{\frac{PT}{N_0}}\left[1 + \frac{2j - (K-1)N}{N}\right]\right), \tag{12}$$

and $Q(\cdot)$ denotes the standard Q-function. Since the bound ρ_K^U is valid for all τ and ϕ, this bound can be used to avoid the issue of how $\rho_K(\tau, \phi)$ depends on the phases and delays. This is the approach followed in [10] and [12]. There may be a large gap between the bound on the bit error probability obtained from (11) and the actual bit error probability evaluated via (9).

[1] There will be one chip $a_{k,j}$ from each interferer which overlaps two adjacent bits $b_{1,i}$ and $b_{1,i+1}$. For $K = 2$, the bit error events are still independent; however, for $K > 2$ adjacent bit error events are not strictly independent. Alternatively, we can consider a correlation receiver which uses only $N - 1$ chips, discarding the last chip of each bit. If $\rho(\tau, \phi)$ is a monotonically decreasing function of N, then the performance of this receiver will upper bound the performance of the correlation receiver which uses N chips. Note also that for $K = 2$, infinite SNR, and τ and ϕ randomly distributed, no bit errors will occur.

We can relate the throughput of both unslotted and slotted systems to the conditional packet error probability $P_E(k)$ given that there are k simultaneous transmissions. Let X_i be the number of packets successfully transmitted during the first i slots. The normalized throughput S is defined as

$$S = \frac{r}{N} \lim_{i \to \infty} E\left[\frac{X_i}{i}\right] \tag{13}$$

where S is normalized for the rate r of the convolutional code, and bandwidth expansion N of the DS/SS system. For a slotted system, the number of active users is constant throughout each packet duration. As a result, the throughput of a slotted system may be given by

$$S = \frac{r}{N} \sum_{k=0}^{\infty} k p_K(k)[1 - P_E(k)] \tag{14}$$

where $P_E(k)$ is the conditional probability of packet error given k users. For an unslotted system we must compute the normalized throughput by taking the expectation over both the number of interferers and packet length.

III. Slotted System Throughput

In this section, we develop a new and tighter bound for the packet error probability and throughput of a slotted DS/SS MA system with hard decision convolutional coding. As discussed in Section II, it is shown in [10] that when all users have equal signal power the worst bit error rate occurs when all multiple-access interferers are phase and chip synchronous, and the probability of error is given by (11).

If there are L bits in a packet, Pursley and Taipale [10] show that the packet error probability $P_E(K)$ for $K = k$ simultaneous transmissions is upper bounded by

$$P_E(k) \leq 1 - [1 - P_\mu(\rho_k(\tau, \phi))]^L \tag{15}$$

where $P_\mu(\rho)$ is the error event probability for the convolutionally coded system with bit error rate ρ. Since the bound ρ_k^U is valid for all τ and ϕ, the bound can be used to avoid the issue of how $\rho_k(\tau, \phi)$ depends on the phases and delays. In [10], the upper bound of (11) is used to evaluate (15), giving an upper bound which is valid for all τ and ϕ

$$P_E(k) \leq P_{E1}(k) = 1 - \left[1 - P_\mu(\rho_k^U)\right]^L \tag{16}$$

and the average packet error probability P_E may be upper bounded by

$$P_E \leq P_{E1} = \sum_{k=0}^{\infty} P_{E1}(k) p_K(k), \tag{17}$$

where we denote by P_{E1} the upper bound that is given in [10] for the probability of error conditioned on K.

We improve on this upper bound $P_{E1}(k)$ in two ways. First, we introduce a new tighter bound on the error event probability of a convolutional code. We call this bound the Improved Union Chernoff Bound. Next, we apply moment space techniques to achieve a tighter upper bound on packet error probability.

The upper bound on $P_\mu(\rho)$ employed in [10] is due to Van de Meerberg [13] and is based on the transfer function $T(D)$ of the convolutional code. This bound can be expressed as

$$P_\mu(\rho) \le \binom{2n_o - 1}{n_o} 2^{-2n_o} \left\{ \frac{1}{2}[T(D) + T(-D)] + \frac{1}{2} D[T(D) - T(-D)] \right\}_{D=2\sqrt{\rho}} \tag{18}$$

where $n_o = \lfloor \frac{d_{\text{free}}+1}{2} \rfloor$ is half the free distance of the convolutional code.

Another tighter upper bound on the probability of an error event maybe derived as follows. The transfer function may be expressed as an infinite series $T(D) = \sum_{i=d_{\text{free}}}^{\infty} t_i D^i$. The probability of an error event may be upper bounded in terms of the coefficients $\{t_i\}$ as

$$P_\mu(\rho) \le \sum_{i=d_{\text{free}}}^{\infty} t_i P_i, \tag{19}$$

where P_i for i odd is

$$P_i = \sum_{j=\frac{i+1}{2}}^{i} \binom{i}{j} \rho^j (1-\rho)^{i-j} \tag{20}$$

and for $i = 2j$ even, $P_{2j} = P_{2j-1}$. We can break the union bound into two parts

$$P_\mu(\rho) \le \sum_{i=d_{\text{free}}}^{M} t_i P_i + \sum_{i=M+1}^{\infty} t_i P_i \tag{21}$$

where M is a large integer. Since $P_i < D^i$

$$P_\mu(\rho) \le \sum_{i=d_{\text{free}}}^{M} t_i P_i + \sum_{i=M+1}^{\infty} t_i D^i \tag{22}$$

$$= \sum_{i=d_{\text{free}}}^{M} t_i P_i - \sum_{i=d_{\text{free}}}^{M} t_i D^i + \sum_{i=d_{\text{free}}}^{\infty} t_i D^i \tag{23}$$

$$\le \sum_{i=d_{\text{free}}}^{M} t_i (P_i - D^i) + T(D) \Bigg|_{D=2\sqrt{\rho(1-\rho)}}, \tag{24}$$

where we have added and subtracted identical terms in (23) before expressing the final bound of (24) in terms of the transfer function $T(D)$. An algorithm for computing the transfer function is given in [7]. Any number of terms of the expansion $\{t_i\}$ can be computed from the transfer function. Furthermore, P_i can be computed in a recursive fashion. We refer to equation (24) as the Improved Union-Chernoff Bound, and we refer to the previous bound on error event probability (18) as the Van de Meerberg bound. An Improved Union-Chernoff bound can also be derived for the probability of bit error.

Our next step is to examine the difference between the average and worst case probability of bit errors. Because of the large gap between the bound on the bit error rate computed from (11) and the actual bit error probability $\bar{\rho}_K$ computed from (9), the new bound we obtain below can be significantly tighter than the worst case bound P_{E1}. Our result is based on the following theorem from [2]:

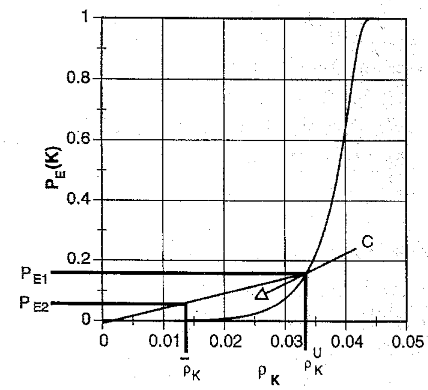

Fig. 3. Illustration of moment space bounding technique.

Moment Space Theorem [2]: Let X be a random variable with a probability distribution function $F_X(x)$ defined over a finite closed interval $I = [a, b]$. Let $g_1(x), g_2(x), \cdots, g_N(x)$ be a set of N continuous functions defined on I. The moment of the random variable X induced by the function $g_i(x)$ is

$$m_i = E[g_i(X)] = \int_I g_i(x) dF_X(x). \tag{25}$$

Now denote the moment space M by

$$M = \{\mathbf{m} = (m_1, \cdots, m_N) \in \Re^N | m_i = \int_I g_i(x) dF_X(x), \quad i = 1, \cdots, N\}, \tag{26}$$

where F_X ranges over the set of probability distributions defined on I, and $\Re^N$ denotes N-dimensional Euclidean space. Then M is a closed, bounded, convex set. Now let H be the convex hull of the curve $(g_1(x), \cdots, g_N(x))$ traced out in $\Re^N$ for $x \in I$. Then

$$H = M. \tag{27}$$

The convex hull of all moment-defining functions traced out in Euclidean N-space contains all of the moments defined by (25).

We take the approach that both $\rho_k(\tau, \phi)$ and $P_E(k)$ are random variables. Fig. 3 shows how these two random variables are related. The packet error probability P_E is some function $f(\rho(\tau, \phi))$. The Moment Space Theorem [2] tells us that the expected value of P_E lies within the intersection of the convex hull generated by $f(\rho(\tau, \phi))$ with the expected value of $\rho(\tau, \phi)$.

The new, tighter bound on $P_E(k)$ is given by the following theorem.

Proposition 1: Consider a slotted DS/SS MA system with $K = k$simultaneous transmissions and let c be a real number such that

$$f(\rho) = 1 - [1 - P_\mu(\rho)]^L \tag{28}$$

is a convex function of ρ on the range $[0, c]$. Then packet error probability may be upper bounded by

$$P_E(k) \le P_{E2}(k) = \begin{cases} \frac{\bar{\rho}_k}{\rho_k^U} P_{E1}(k), & \rho_k^U \in [0, c) \\ 1, & \rho_k^U \notin [0, c). \end{cases} \tag{29}$$

Proof: It is obvious that packet error probability is always upper bounded by 1. For $\rho_k^U \in [0, c]$, the random variable $\rho_k(\tau, \phi)$ lies in the interval $[0, \rho_k^U]$ and has expected value $\bar{\rho}_k$. From [10] the packet error probability is upper bounded by a function $f(\rho_k(\tau, \phi))$, where $f(\rho) = 1 - [1 - P_\mu(\rho)]^L$. Let H be the convex hull in $\Re^2$ of all points in the set $\{(x_1, x_2) : x_1 = \rho_k(\tau, \phi), x_2 = f(\rho_k(\tau, \phi))\}$. Then the Moment Space Theorem [2] requires that $P_E(k) \leq E[f(\rho_k(\tau, \phi)]$ be upper bounded by a least one point in the intersection of H and $x_1 = \bar{\rho}_k$. Since $f(\rho)$ is a convex function of ρ for $\rho_k^U \in [0, c]$, the convex hull H is upper bounded by a straight line from the point $(0, 0)$ to the point $(\rho_k^U, f(\rho_k^U))$. Therefore the intersection of H with moment $\bar{\rho}_k$ consists of $\{(x_1, x_2) : x_1 = \bar{\rho}_k, x_2 \in [f(\bar{\rho}_k)), \frac{\bar{\rho}_k}{\rho_k^U} P_{E1}(k)]\}$. The theorem follows. □

We call the resulting bound on packet error probability$P_{E2}(k)$. The restriction that $f(\rho)$ be a convex function for $\rho \in [0, c)$ is mild. In most cases $f(\rho)$ is convex for all ρ of interest. Normalized throughput for the slotted case is given by (14). Truncating the summation in (14) results in a lower bound on achievable throughput, as does using an upper bound on $P_E(k)$. For the numerical results presented here, we have truncated all terms for which $P_E(k) > 0.99$. We call the bounds on throughput obtained by using $P_{E1}(k)$ and $P_{E2}(k)$, S_1 and S_2 respectively.

IV. Unslotted System Throughput

In an unslotted system, the number of users varies with time. Let$K(i)$ be the number of users transmitting during the ith data bit of the packet. We assume that $K(i) \leq K_{\max}$ for all i and some value of $K_{\max}$. If $K(i)$ does exceed $K_{\max}$, then we will generate a lower bound on achievable throughput. Storey and Tobagi [12] show that the bound of (11) can be extended to the case of an unslotted system. If we let L_k equal the number of data bit intervals i for which $K(i) = k$ (implying $\sum_k L_k = L$), then we have [12]

$$P_E \leq 1 - \prod_{i=1}^{L} [1 - P_\mu(\rho_{K(i)}(\tau, \phi))] \tag{30}$$

$$\approx 1 - \prod_{k=1}^{K_{max}} [1 - P_\mu(\rho_k(\tau, \phi))]^{L_k} \tag{31}$$

$$\leq P_{E1} = 1 - \prod_{k=1}^{K_{max}} \left[1 - P_\mu(\rho_k^U)\right]^{L_k}. \tag{32}$$

We have assumed that the number of interferers $K(i) = k$ is constant for the duration of an error event. The approximation in (31) results from this assumption. In [12] this assumption, called the "memoryless approximation," is shown to be valid through simulations.

We obtain a tighter bound on packet error probability by means of the following proposition.

Proposition 2: Consider an unslotted DS/SS MA system with convolutional coding, and hard-decision decoding. Let $\{L_1 = l_1, L_2 = l_2, \cdots, L_{K_{\max}} = l_{K_{max}}\}$,$R_k = \frac{\bar{\rho}_k}{\rho_k^U}$ for $k \in \{2, \cdots, K_{\max}\}$, $R_{\max} = \max\{\mathrm{R}_2, \cdots, \mathrm{R}_{\mathrm{K}_{\max}}\}$, and let c_k be a set of real numbers such that

$$g(\rho) = g(\rho_2, \cdots, \rho_{K_{\max}}) = 1 - \prod_{k=1}^{K_{\max}} [1 - P_\mu(\rho_k)]^{l_k} \tag{33}$$

is a convex function of each ρ_k for $\rho_k \in [0, c_k], k \in \{2, \cdots, K_{\max}\}$. Then the packet error probability is upper bounded by

$$P_E \leq P_{E2} = \begin{cases} R_{\max}[P_{E1} - P_E(1)] + P_E(1), & \rho_k^U \in [0, c_k] \forall k \\ 1, & \text{else.} \end{cases} \tag{34}$$

In practice, $\rho_K^U \in [0, c_k]$ for almost all cases of interest.

Proof: It is obvious that packet error probability is no more than1. We consider the case when $g(\rho) = g(\rho_2, \cdots, \rho_{K_{\max}})$ is a convex function for each $\rho_k \in [0, c_k], k \in \{2, \cdots, K_{\max}\}$. First suppose that $l_1 = 0$ ($K(i) \geq 2 \ \forall i$). Let $\rho(\tau, \phi) = (\rho_2(\tau, \phi), \cdots, \rho_{K_{\max}}(\tau, \phi))$ be a random vector which lies in the region $V = \{(x_2, \cdots, x_{K_{\max}}) : x_k \in [0, \rho_k^U]\}$ in $\Re^{K_{\max}-1}$. From (31), P_E is upper bounded by a function $g(\rho(\tau, \phi))$, where $g(\rho) = 1 - \prod_{k=1}^{K_{\max}} [1 - P_\mu(\rho_k)]^{l_k}$. Let H be the convex hull in $\Re^{K_{\max}}$ of all points in the set $\{(x_1, \cdots, x_{K_{\max}}) : x_1 = g(\rho), x_k = \rho_k$ for $k = 2, \cdots, K_{\max}\}$. Then the moment space theorem [2] requires that $P_E = E[g(\rho)]$ lie in the intersection of H and $\{(x_2, \cdots, x_{K_{\max}}) : x_k = \bar{\rho}_k$ for $k = 2, \cdots, K_{\max}\}$.

Next, we show that if P_E lies in this intersection, then $P_E \leq R_{\max} P_{E1}$. Recall that $g(\rho(\tau, \phi))$ is a convex function of each variable $\rho_K(\tau, \phi)$, and that $g(\mathbf{0}) = 0$ and $g(\rho_2^U, \cdots, \rho_{K_{\max}}^U) = P_{E1}$ is the maximum value of $g(\rho)$ for $\rho \in V$. Now consider any point $(\acute{g}, \bar{\rho}_2, \cdots, \bar{\rho}_{K_{\max}}) \in H$ and draw a line from the origin to that point. The line is described by the equation $\acute{g} = c_2 \bar{\rho}_2 + \cdots + c_{K_{\max}} \bar{\rho}_{K_{\max}}$ for some set of coefficients $\{c_2, \cdots, c_{K_{\max}}\}$. Now consider the point $c_2 \rho_2^U + \cdots + c_{K_{\max}} \rho_{K_{\max}}^U \leq P_{E1}$. This implies that for some value of $\frac{\acute{g}}{P_{E1}} \leq \frac{\bar{\rho}_k}{\rho_k^U}$ for at least one $k \in \{2, \cdots, K_{\max}\}$. It follows that P_E is bounded by $P_E \leq R_{\max} P_{E1}$.

Now relax the condition that $l_1 = 0$. Note that when no multiple access interference is present, the occurrence of a bit error in any interval is independent of the occurrence of a bit error in any other interval. Therefore the probability that an error occurs when no interferers are present is given by $1 - [1 - P_\mu(\bar{\rho}_1)]^{l_1} \leq 1 - [1 - P_\mu(\bar{\rho}_1)]^L = P_E(1)$. The dependence of P_E on the function $g(\cdot)$ is exactly the same as before, except the bound must be shifted up by $P_E(1)$. Proposition 2 follows. □

Let $P_{E,l}$ be the probability of error in an unslotted system for a packet of length l, and let $p_L(l)$ be the probability mass function of packet length L. Given the initial distribution of users and the rates at which packet transmissions are initiated and completed, we can use a Markov chain to compute the tighter bound on packet error probability $P_{E2,l}$ for each l from (34). The overall packet error probability is then

$$P_E \leq P_{E2} = \sum_{l=1}^{\infty} p_L(l) P_{E2,l}. \tag{35}$$

TABLE I
A COMPARISON OF $P_E(K)$ BOUNDS FOR SLOTTED DS/SS MA (Rate 1/2, CONSTRAINT LENGTH 7 CONVOLUTIONAL CODE, $N = 31$)

SNR (dB)	K	P_{E1}	P_{E2}	P_{E3}
10	3	2.26e-5	5.24e-7	5.07e-7
	5	4.05e-3	1.77e-4	1.59e-4
12	3	1.47e-7	1.56e-9	1.54e-9
	5	2.96e-4	7.51e-6	7.11e-6
15	3	1.07e-10	5.51e-13	5.48e-13
	5	1.44e-5	2.10e-7	2.04e-7

We can truncate (35) to a finite number of terms by assuming that $P_{E2,l} = 1$ for large l. The normalized throughput S can be computed from

$$S = \frac{r}{N}\sum_{l=1}^{\infty} p_L(l)\frac{l}{\bar{L}}[1 - P_{E,l}]. \tag{36}$$

Truncating (36) to a finite number of terms will result in a lower bound on the achievable S.

V. NUMERICAL RESULTS

We have used the new bounds to compute packet error probability and throughput for a DS/SS MA system employing the standard rate 1/2, constraint length 7 convolutional code with free distance 10. The transfer function of this code is given in [7]. For this convolutional code with $L = 1000$, the function $f(\rho)$ is convex for $\rho \in [0, 0.04097]$ if the bound of Van de Meerberg is used for $P_\mu(\rho)$, and convex for $\rho \in [0, 0.04330]$ if the Improved Union–Chernoff bound is used.

Table I compares the several bounds on packet error probability. The bound from [10] is denoted P_{E1}. The new bound obtained from applying Proposition 1 with Van de Meerbergs's bound on error event probability $P_\mu(\rho)$ is denoted P_{E2}, while the bound obtained by using the Improved Union–Chernoff on $P_\mu(\rho)$ is denoted P_{E3}. Results are shown for $K = 3$ and 5, $N = 31$ and signal-to-noise ratio (SNR) equal to 10, 12 and 15 dB. In each case, P_{E2} is tighter than P_{E1} by over an order of magnitude. The bound on the packet error probability that results from the Improved Union–Chernoff bound on $P_\mu(\rho)$ is a modest improvement on P_{E2}.

Next, we consider a system with $N = 63$ and$SNR = 8$ dB, and use only the Van de Meerberg bound on $P_\mu(\rho)$. Once again, a rate 1/2, constraint length 7 convolutional code is used. Fig. 4 compares the two bounds P_{E1} and P_{E2} for a slotted system with $L = 1000$ and K ranging from 2 to 12. Once again, the new bound is significantly tighter over a wide range of multiple access interference. Fig. 5 plots the normalized throughput S versus the offered traffic K for this same slotted system. The bounds S_1 and S_2 are obtained by substituting P_{E1} and P_{E2} respectively into (14). For lightly loaded systems, the two bounds on throughput coincide, because packet errors have an insignificant effect on system throughput for such systems. For more heavily loaded systems, there is a large gap in the throughput predicted by the two bounds.

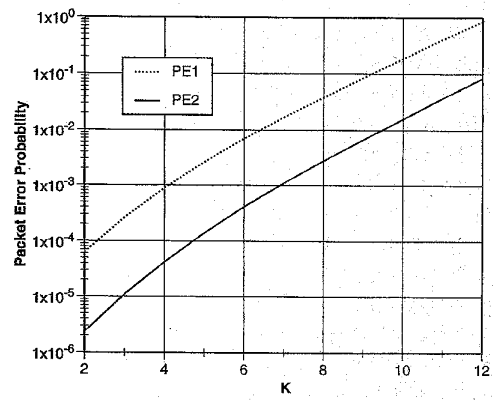

Fig. 4. Packet error probability versus number of active users for slotted DS/SSMA.

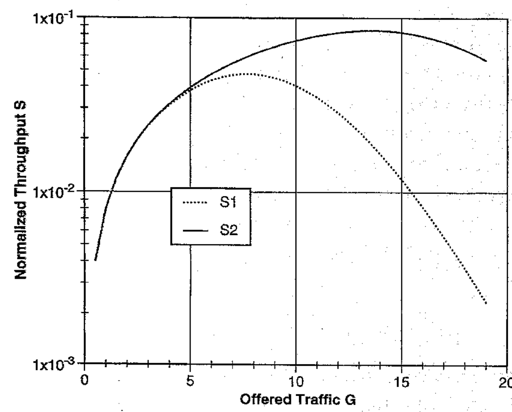

Fig. 5. Normalized throughput versus offered traffic for slotted DS/SSMA.

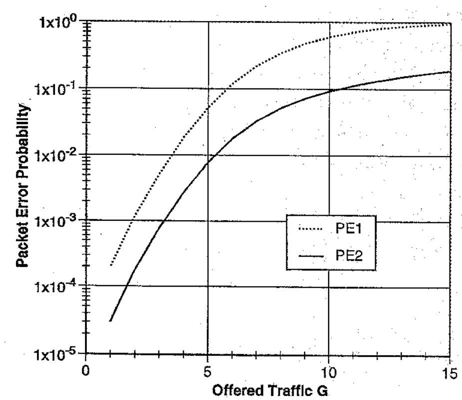

Fig. 6. Packet error probability versus offered traffic for unslotted DS/SSMA.

Figs. 6 and 7 report packet error probability and normalized throughput for an unslotted system with $\bar{L} = 1000$. Once again the new bounds P_{E2} and S_2 are significantly tighter than the bounds P_{E1} and S_2.

Next, we explore the relationship between the choice of error correcting code and system performance. We apply the techniques of the previous section to evaluate the throughputs S_2 for a large number of different code rates. For the comparison, we will compare the results for our bounding technique with an approximation technique described in [11]. We will denote this approximation to throughput as S_{app}. For the results reported here, we examined a slotted DS/SS MA

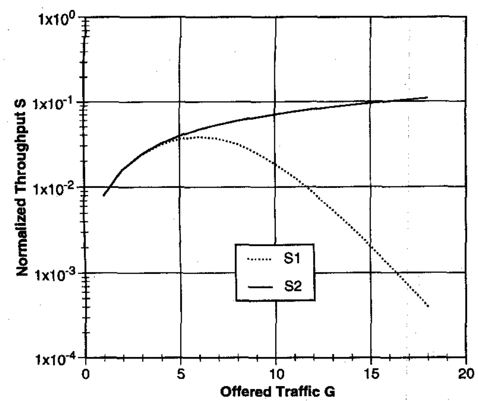

Fig. 7. Normalized throughput versus offered traffic for unslotted DS/SSMA.

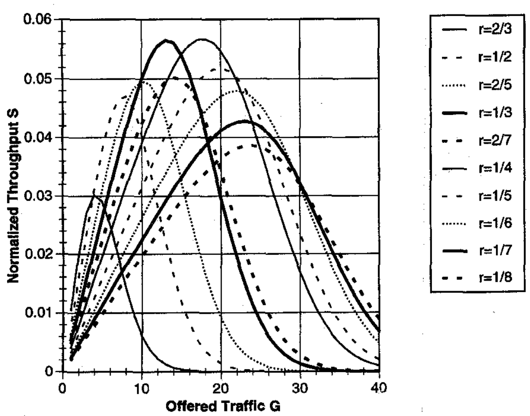

Fig. 8. Normalized throughput S_2 versus offered traffic G for several code rates using the lower bounding technique.

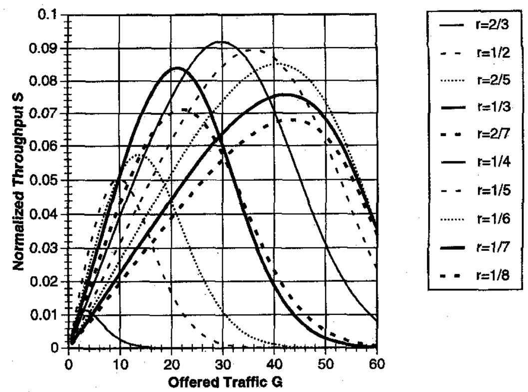

Fig. 9. Normalized throughput S_{app} versus offered traffic G for several code rates using the approximation technique.

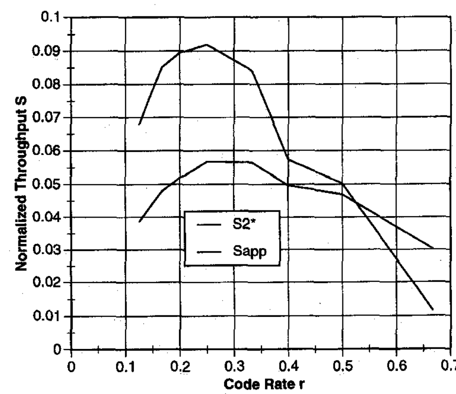

Fig. 10. Maximum normalized throughput $S^*(r)$ as a function of code rate r for both systems.

system with processing gain $N = 63$ and the number of users ranging from $K = 0$ through $K = 60$. Once again, we assume that $E_b/N_o = 8$ dB, and that each packet consists of $L = 1000$ information bits to which coding is added.

We examined convolutional codes with rates ranging from $r = 1/8$ to $2/3$. In order to insure that the codes were of comparable complexity, we held the number of states times the number of branches emerging from each state constant at 64 for each code. A rate $1/2$ constraint length 6 code served as the baseline system.

Figs. 8 and 9 plot the throughput S versus offered traffic G for both the lower bound on throughput and the approximation technique, respectively. For each different code, the throughput increases to some maximum level as G increases, and then decays once the error correcting capability of the code is exceeded. The peak throughput occurs at a higher traffic level for the more powerful (lower rate) codes.

With one exception, the lower bound reports smaller values for throughput than the approximation technique. The difference in reported throughput becomes quite large for the condition of high offered load. In practice, this large throughput is obtained at the price of frequent packet errors and retransmissions. Fig. 10 plots the maximum throughput $S^*(r)$ of any system with code rate r. Note that both techniques predict that the system should attain its maximum level of throughput when $r = 1/4$. In the case of $r = 2/3$ the approximation is actually less than the lower bound.

Fig. 11 plots throughput versus offered traffic, assuming that in each case, the code rate r is selected to optimize system performance for the value of G. For low levels of offered traffic, the lower bound and approximation track one another closely. For large G, the approximation predicts dramatically larger throughput levels. However, the optimum codes rates $r^*(G)$ predicted by the two techniques for a given G remain remarkably close to one another throughout the entire range of G. These optimum code rates are displayed in Table II. As G increases, the predicted optimum codes rates are reasonably close, even though the predicted throughput is dramatically different. The lower bound method suggests only a slightly more conservative choice for large G. This implies that the system performance is relatively robust with respect to the technique used to select the code rate.

VI. Conclusions

In this paper, we have derived new, tighter bounds on the packet error probability of a DS/SS system with convolutional coding which improve on the bounds of [10] and [12]. These bounds were based on the use of an Improved Union–Chernoff Bound for the error event probability of a convolutional code and on the application of moment space techniques. Bounds were derived for both the cases of slotted and unslotted

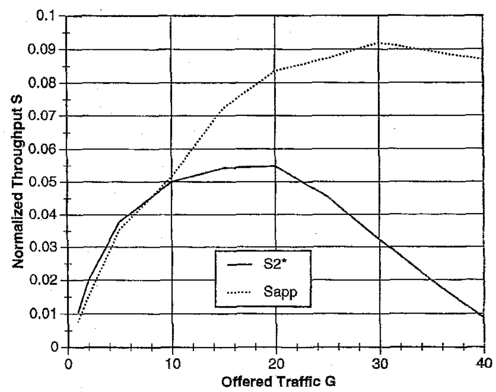

Fig. 11. Maximum normalized throughput using rate $r^*(G)$ as a function of offered traffic G for both systems.

TABLE II
OPTIMAL CHOICE OF CODE RATES PREDICTED BY THE TWO TECHNIQUES FOR SELECTED VALUES OF G

G	$r*(G)$ predicted by bound	$r*(G)$ predicted by approximation
1.0	2/3	1/2
2.0	2/3	1/2
5.0	1/2	1/2
10.0	2/5	2/5
15.0	1/3	1/3
20.0	1/4	1/3
25.0	1/5	1/4
30.0	1/6	1/4
35.0	1/8	1/5
40.0	1/8	1/6

systems. In each case, the bounds were approximately an order of magnitude tighter than the previous bounds.

We have used our bounds to determine the optimum code rate in order to maximize throughput for a system with convolutional coding. Results indicate that the optimal code rates indicated by this new bounding technique agree well with the optimal code rates which are indicated by approximate techniques even though the two techniques produce dissimilar results for throughput.

Overall, we find that the assumption of phase and chip synchronous interference is extremely pessimistic. As a result, it may be possible to achieve higher levels of throughput for a DS/SSMA system than has been previously reported. Furthermore, we conjecture that future research may lead to still tighter bounds than the ones reported here.

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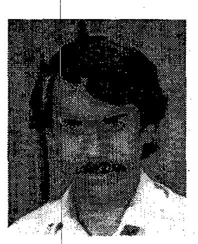

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