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Decoding Algorithms with Input Quantization and Maximum Error Correction Capability

Amer Hassan, *Senior Member, IEEE*,
and Wayne Stark, *Senior Member, IEEE*

***Abstract*— In this correspondence, decoding that uses soft-decision information but with multiple low-complexity decoders are investigated. These decoders correct only errors and erasures. The structure of the receiver consists of a bank of z demodulators followed by errors- and erasures-correcting decoders operating in parallel. Each demodulator has a threshold for determining when to erase a given symbol. We assign a cost $f(\theta)$ to the noise for causing an erasure when the receiver uses a particular threshold θ and a (larger) cost $f(\bar{\theta})$ for causing an error. The goal in designing the receiver is to choose the thresholds to maximize the noise cost which is necessary to cause a decoding error. We demonstrate that the above formulation is solvable for many channels including the M-ary input–output channel, the additive channel with coherent demodulation, and an additive channel with orthogonal modulation and noncoherent demodulation. Then we show that the maximum worst case error-correcting capability of the parallel decoding algorithms is the same as the maximum worst case error-correcting capability of a correlation decoder with the same number of quantization regions.**

***Index Terms*—Soft-decision decoding, parallel decoding, concatenated codes, quantization, correlation decoding, communications gain.**

I. Introduction

Parallel decoding (or equivalently, multiple decoding) of a noise-corrupted codeword is of considerable interest in improving the error-correcting capability of a decoder. Parallel decoding for concatenated codes was investigated by Zyablov [3] for M-ary input–output channels with the Hamming distance as the cost function. If $d_{H,1}$ and $d_{H,2}$ are the minimum Hamming distances of the inner and outer codes, respectively, then Zyablov's decoder corrects all error patterns with weight $\lfloor (d_{H,1}d_{H,2} - 1)/2 \rfloor$ or less. This is the largest error correction possible.

For channels having infinite output alphabets, parallel decoding algorithms addressed in the literature can be applied to an additive noise channel with output limited to the interval $[-1, +1]$. For this channel Forney [1] proposed the generalized minimum distance (GMD) decoding algorithm. Kovalev in [2] proposed modifications of decoding with respect to the GMD metric that incorporates additional complexity constraints. Also, an algorithm was proposed by Dumer *et. al.* [5] for binary concatenated decoding with respect to the generalized minimum distance. A brief overview of the above algorithms and others can be found in [4].

In [4], parallel decoding for an additive channel was investigated with the Euclidean distance as the basic metric instead of Hamming distance or GMD. A closed-form expression was derived for the minimum Euclidean length of a noise vector to cause an error, as a function of the number of branches in the parallel structure and the thresholds employed. Also, a simple algorithm to determine the

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A. Hassan is with Ericsson Mobile Communications Inc., Research Triangle Park, NC 27709 USA.

W. Stark is with the Electrical Engineering and Computer Systems Department, The University of Michigan, Ann Arbor, MI 48109 USA.

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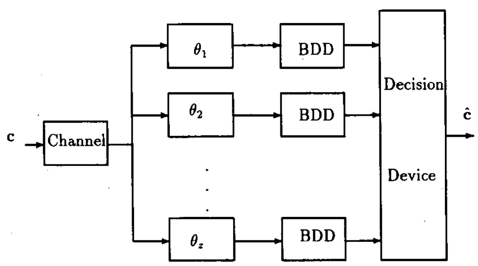

Fig. 1. Parallel decoding with z distinct inner decoders or demodulators and the same outer-bounded distance decoders (BDD).

optimal thresholds and the resulting minimum-length noise vector that will cause an error was found.

In this correspondence, the general formulation for parallel decoding that includes several channel models and decoding structures is developed. The structure of the receiver consists of a bank of z demodulators followed by errors-and-erasures correcting decoders operating in parallel. Each demodulator has a threshold θ that determines an erasure region; we then assign a cost $f(\theta)$ to the noise for causing an erasure and a (larger) cost $f(\overline{\theta})$ for causing an error. Since we are concerned with worst case noise, throughout the correspondence the noise considered is an intelligent jammer. The goal in designing the receiver is to choose the thresholds to maximize the jammer cost necessary to cause a decoding error. We demonstrate that the above formulation is solvable for many channels including the simple M-ary input–output channel with the Hamming distance as the cost function, the additive channel where the cost function corresponds to Euclidean distance, and a noncoherent channel with ratio-threshold-like decision rules and difference metric decision rules. We describe the solutions for these channels and find the worst case noise correctable by the decoder.

Secondly, we evaluate the error-correcting capability of a correlation decoder. For this decoder, the demodulator output is quantized to one of z levels. A metric m_i is assigned to region (level) i, which is used in the correlation. The optimal parallel decoding algorithms are shown to be equivalent, in terms of error-correcting capability, to a correlation decoder with optimal thresholds and metrics. The quantization levels and constraints on the metrics are the solution to an optimization "max-min" problem.

The correspondence is organized as follows. A general formulation for parallel decoding is discussed in Section II. The optimal thresholds and error-correcting capability is shown to be the solution of a set of nonlinear equations. This solution is applied to particular cases of interest where the gain obtained by using parallel decoding as compared with the best single-branch decoder is demonstrated. In Section III, a different decoding structure, namely correlation decoding with quantized regions, is analyzed. In Section IV conclusions are stated.

II. Parallel Decoding

In parallel decoding, the channel output is processed by z branches; each branch consists of a demodulator/inner decoder connected to an outer decoder. The ith demodulator/decoder is characterized by a threshold θ_i for deciding whether to erase or to output its best estimate to the outer decoder; the input to the outer decoder is then an erasure, a correct estimate, or an erroneous symbol. Then z-identical outer bounded-distance decoders (one for each branch) are used to correct the maximum number of errors and erasures, as shown in Fig. 1.

The receiver, therefore, produces z candidate estimates of the transmitted codeword in which the most likely vector is selected. When designed properly, each threshold is at least nearly optimum for a subclass of channels. It is then of interest to find the best decision regions for erasing such that a given performance measure is optimized, in the presence of jamming. The jammer is able to distort the signal at will but at a certain cost.

Let θ_i parameterize the threshold for erasing for the ith demodulator (inner decoder). The cost to the jammer of causing an erasure is $f(\theta_i)$. The jammer incurs a larger cost $f(\overline{\theta}_i)$ for causing an error to the nearest code symbol. The above communication system is then characterized by the following game:

Communicator's Game: The communicator wants to choose the thresholds $\theta_1, \cdots, \theta_z$ to maximize the minimum cost necessary for a jammer to cause the overall decoding system to err (not decode to the correct codeword).

Jammer's Game: The jammer wants to minimize the cost needed to force the communicator to cause an error no matter what thresholds are used.

We solve this game for arbitrary $f(\theta)$. We also obtain the particular "worst case" noise.

• *Example*

Consider a concatenated code over an M-ary channel with an inner code minimum distance $d_{H,1} = 14$ and an outer code minimum distance $d_{H,2} = 5$. Thus the overall code has minimum Hamming distance 70 (actually this is a lower bound on the overall Hamming distance), and the error correction capability is 34 channel errors.

Simplest Decoding Algorithm: The inner decoder corrects the maximum number of errors possible, that is up to six channel errors, and guesses otherwise. The outer decoder also corrects the maximum number of two errors or less.

Error-Correcting Capability: This outer decoder can correct up to two symbol errors. Each of these could be caused by as few as seven channel errors. The smallest "weight" vector that is not correctable consists of three vectors of weight 7. So the error correcting capability of the decoder is $21 - 1 = 20$.

More complex decoder: If an inner received word is within Hamming distance 4 or less of a codeword, the output of the inner decoder is that codeword. Otherwise, the inner outputs decoder is an erasure. The outer decoder is a bounded distance decoder that corrects all patterns of e errors and τ erasures as long as $2e + \tau$ is strictly less than 5.

The above algorithm can correct 24 errors. (The vector which is distance 5 from the transmitted codeword in each of the five places that two codewords differ is not correctable.)

We can actually do better if extra complexity can be tolerated. Consider two branches at the receiver, with the first inner decoder characterized by θ_1 and the second inner decoder characterized by θ_2.

The optimal choices can be shown to be $\theta_1 = 3$, $\theta_2 = 6$. This decoder can correct 29 errors which is a substantial increase in the number of errors corrected as compared to the single-branch case.

For four branches such that inner decoder 1 is used for single error correction, decoder 2 for double error correction, decoder 3 is used to correct four errors, and decoder 4 is used to correct five errors, then all the errors promised by the overall distance 70 is realized; that is, all error patterns with weight 34 or less can be corrected by the decoding algorithm.

Assume that the kth threshold results in e_k errors and τ_k erasures. Assume also that if decoder k erases a particular symbol then decoder l will also erase that symbol if $l \leq k$. That is, the regions for decoding a particular symbol get progressively larger and the erasure regions get smaller. Also, assume that the only way the overall decoder will err is if all of the individual decoders err.

For a fixed number of errors and erasures, the noise vector with the least cost will be in the direction of the closest codeword. The

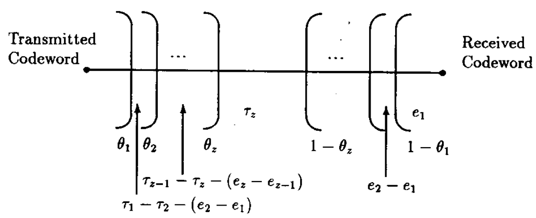

Fig. 2. Decoding regions for demodulators or inner decoders.

received vector consists of n received symbols (or inner codewords). If there are e_1 received vectors that cause the first decoder to put out an incorrect codeword the received vectors cost must be at least $f(\overline{\theta}_1)$ with equality if the noise vectors in all of these (inner) received vectors are in the direction of the closest (inner) codeword from the transmitted codeword. If there are e_2 received vectors that cause the second inner decoder to produce an incorrect codeword then e_1 of these will also cause the first decoder to produce an incorrect codeword. Thus $e_2 - e_1$ received vectors will produce erroneous codewords from decoder 2 and will be erased by decoder 1. These received vectors cost is at least distance $f(\overline{\theta}_2)$ with equality if the noise vectors are directed as before. For the zth decoder there will be $e_z - e_{z-1}$ received vectors that are decoded incorrectly that are erased by the other decoders. These noise vectors must cost at least $f(\overline{\theta}_z)$. There are τ_z received vectors that are erased by the zth decoder. The cost incurred by the jammer is at least $f(\overline{\theta}_z)$. There are τ_{z-1} vectors that cause decoder $z-1$ to produce an erasure. Of these τ_z are also erased by the zth decoder and $e_z - e_{z-1}$ cause an error in the zth decoder but not in decoder $z-1$ and have already been taken into account. The remaining $\tau_{z-1} - \tau_z - (e_z - e_{z-1})$ must have a cost of $f(\overline{\theta}_{z-1})$. This reasoning is demonstrated in Fig. 2. Continuing in this line of reasoning, the minimum jammer's cost needed to cause $\boldsymbol{\tau}$ erasures and $\boldsymbol{e}$ errors and for arbitrary thresholds $\boldsymbol{\theta} = \{\theta_1, \cdots, \theta_z\}$ is given in the following equation:

$$\begin{aligned}\gamma(\boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{e}) &= \tau_z f(\theta_z) + (\tau_{z-1} - \tau_z - (e_z - e_{z-1}))f(\theta_{z-1}) \\ &\quad + \cdots + (\tau_1 - \tau_2 - (e_2 - e_1))f(\theta_1) \\ &\quad + (e_z - e_{z-1})f(\overline{\theta}_z) + (e_{z-1} - e_{z-2})f(\overline{\theta}_{z-1}) \\ &\quad + \cdots + (e_1)f(\overline{\theta}_1)\end{aligned}$$

where $\boldsymbol{\tau} = \{\tau_1, \cdots, \tau_z\}$ and $\boldsymbol{e} = \{e_1, \cdots, e_z\}$.

The problem is then to solve the following max-min:

$$\max_{\boldsymbol{\theta}} \min_{\boldsymbol{\tau}, \boldsymbol{e}} \gamma(\boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{e})$$

where the minimization is over all vectors $\boldsymbol{\tau}, \boldsymbol{e}$ that cause each of the decoders to not decode correctly. That is

$$\begin{aligned}2e_i + \tau_i &\geq d_{H,2} \\ \tau_i &\geq \tau_{i+1} \\ e_i + \tau_i &\leq n, \qquad i = 1, 2, \cdots, z.\end{aligned}$$

It can be shown that the above problem reduces to the following [4]:

$$\max_{\boldsymbol{\theta}} \min_{1 \leq k \leq z+1} f(\overline{\theta}_k) + f(\theta_{k-1}).$$

We introduce some notation needed in the solution of this game. Let $f(\theta_0) = 0$; moreover, let

$$\begin{aligned}f(\theta_{z+1}) &= f(\overline{\theta}_z) \\ f(\overline{\theta}_{z+1}) &= f(\theta_z).\end{aligned}$$

The solution of the game can be evaluated as in [4] and satisfies the following set of equations:

$$\begin{aligned}f(\overline{\theta}_k) + f(\theta_{k-1}) &= \alpha, \qquad k = 1, 2, \cdots, z. \\ 2f(\overline{\theta}_z) &= \alpha.\end{aligned} \tag{1}$$

The worst case noise vector is not in general unique. One particular worst case noise vector can be described as follows. First, in a single codeword with distance $d_{H,2}$ at most $d_{H,2}$ components of the noise vector are nonzero. The simplest strategy is to force erasures in all the decoders by placing the received vector at distance θ_z from the transmitted codeword. In this case, the jammer would have cost $d_{H,2} f(\theta_z)$. Now it is easy to see that we can modify the jamming strategy without changing the cost. Any pair of noise vectors can be moved without changing the cost (or the fact that the number of errors and erasures cause the decoder to fail) in the following way. Move one of the noise vectors to threshold $\overline{\theta}_k$ and move the other to threshold θ_{k-1}. Because of the above equations we have not changed the cost to the jammer. We have changed the number of errors and erasures for the different decoders. Decoders $1, \cdots, (k-1)$ have the same number of errors and erasures. Decoders $k, \cdots, z$ now have two less erasures but one more error. Thus $2e_k + \tau_k$ remains the same. There are many different jamming strategies that all yield the same cost and cause the decoders to fail. Jammers that have a fractional number of components at a given length may also satisfy the above equations but are not of interest.

The resulting minimum cost for the jammer to cause an error is given by

$$\alpha d_{H,2}/2$$

where $d_{H,2}$ is the Hamming distance of the (outer) code. This solution for the discrete "concatenated code" version is valid if there is no rounding in the solution of the above equations (i.e., the θ_i that solve the equations are integers).

Next we apply the solution of the above game to the following cases of interest:

1) Two-stage decoding for concatenated codes used on an M-ary input/output channel. Cost is taken to be the number of errors caused by the channel (i.e., made by the decoder).
2) Additive noise channel, either M-ary demodulation or two-stage decoding for block concatenated codes used on an additive channel. Cost is the Euclidean distance of jamming noise.
3) Noncoherent Demodulation of M-ary FSK. Cost is the worst case noise length added to the signal in order to cause an error.

A. Hamming Distance

Consider a concatenated code with an inner block (or convolutional) code. The ith inner decoder corrects all error patterns with Hamming weight **strictly** less than θ_i, and the decoders for the outer code are the same bounded-distance decoders that correct errors and erasures. The maximum number of errors always corrected by the overall decoder is then of interest. The cost function is as follows:

$$\begin{aligned}f(\theta_i) &= \theta_i \\ f(\overline{\theta}_i) &= d - \theta_i + 1\end{aligned}$$

where d is the minimum Hamming distance of the inner code.

- The optimal threshold for $z = 1$ and the error-correcting capability are found by solving the following two equations:

$$\begin{aligned}f(\overline{\theta}_1) + f(\theta_0) = \alpha &\rightarrow d - \theta_1 + 1 = \alpha \\ f(\overline{\theta}_2) + f(\theta_1) = \alpha &\rightarrow 2\theta_1 = \alpha.\end{aligned}$$

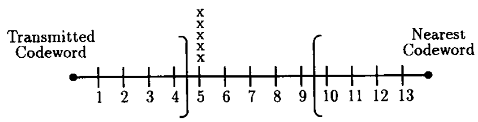

Fig. 3. An example of concatenated code. Inner code distance = 14, outer code distance 5. Shortest vector not correctable has length 25.

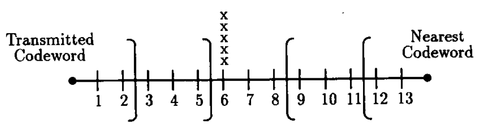

Fig. 4. An example of concatenated code. Inner code distance = 14, outer code distance 5. Shortest vector not correctable has length 30.

Solving the above two equations gives the optimal threshold setting for the inner decoder and a single branch

$$\theta_1 = \left\lfloor \frac{d+1}{3} \right\rfloor.$$

With no roundoff in the above equation (i.e., $d+1 \equiv 0 \bmod 3$)

$$\alpha = \frac{2(d+1)}{3}.$$

The shortest vector not correctable has length

$$\gamma = \frac{d_{H,2}}{2} \frac{2(d+1)}{3}.$$

The resulting correctable noise length is $\alpha d_{H,2}/2 - 1$. For $d = 14$ and $d_{H,2} = 5, \theta = 5$ (inner decoder corrects four or less errors) and $\alpha d_{H,2}/2 - 1 = 24$. The shortest vector not correctable is shown in Fig. 3. It consists of five inner codewords with five errors each, in the direction of the nearest codeword.

As mentioned earlier, a noise vector with three components of length 5 and one of length 10 would also have the minimum length as would a noise vector with two components of length 10 and one of length 5.

- $z = 2$ results in the following optimal thresholds:

$$\theta_1 = \left\lfloor \frac{d+1}{5} \right\rfloor$$
$$\theta_2 = \left\lfloor \frac{2d+2}{5} \right\rfloor.$$

If $d+1 \equiv 0 \bmod 5$, the optimal value for α is

$$\alpha = \frac{4(d+1)}{5}.$$

Thus the shortest vector not correctable has length

$$\alpha = \frac{d_{H,2}}{2} \frac{4(d+1)}{5}.$$

For the example considered, the shortest vector not correctable by the code has weight 30 and is shown in Fig. 4. Fig. 5 shows the shortest vector not correctable for a decoder with nonoptimal thresholds; it has a weight 29 (less than 30 for optimal thresholds).

It is interesting to note that if fractional errors are allowed, the optimal jammer is a two-level jammer; for the above example we would have 2.5 codewords at distance 3 and 2.5 codewords at distance 9 (nothing at distance 7), which will cause a decoder failure. The

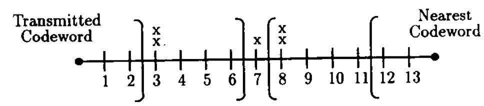

Fig. 5. An example of concatenated code but not optimal thresholds. Inner code distance = 14; outer code distance 5. Decoder 1: five erasures (not correctable). Decoder 2: two errors and one erasure (not correctable). Noise vector length = $16 + 7 + 6 = 29 < 30$.

noise length in this case is 30. In general

$$\begin{aligned} d - \theta_1 + 1 &= \alpha \\ d - \theta_2 + 1 + \theta_1 &= \alpha \\ d - \theta_3 + 1 + \theta_2 &= \alpha \\ &\vdots \\ d - \theta_z + 1 + \theta_{z-1} &= \alpha \\ 2\theta_z &= \alpha. \end{aligned}$$

Adding all the above equations gives

$$\theta_z = \left\lfloor \frac{z}{2z+1}(d+1) \right\rfloor.$$

Solving for the rest of the thresholds we have

$$\theta_k = \left\lfloor \frac{k}{2z+1}(d+1) \right\rfloor.$$

With no roundoffs

$$\alpha = \frac{2z}{2z+1}(d+1).$$

Thus the shortest vector not correctable has length

$$\alpha = \frac{d_{H,2}}{2} \frac{2z}{2z+1}(d+1).$$

The above value for α holds in the case of no roundoffs of the thresholds (i.e., $(d+1) \equiv 0 \bmod z$). Actually, for finite d we need only a finite number of decoders to achieve the maximum possible correcting capability. These results are the same as obtained by Zyablov [3].

B. Euclidean Distance

Consider a concatenated code with an inner block code that corrects errors with respect to the Euclidean distance and an outer code that corrects errors and erasures. In this case

$$\begin{aligned} f(\theta_i) &= \theta_i^2 \\ f(\overline{\theta}_i) &= (d_{E,1} - \theta_i)^2 \end{aligned}$$

where θ_i is the inner decoder's correctable Euclidean distance, and $d_{E,1}$ is the minimum Euclidean distance of the inner code. As a baseline, if an inner decoder is chosen to correct the longest Euclidean noise, the concatenated code is able to correct all noise lengths less than $d_{H,2} d_{E,1}^2/8$ (for $d_{H,2}$ even). It can be shown that for optimal thresholds, and fixed z, the gain is $4\alpha_z$. We assume throughout that $d_{E,1} = 1$.

- $z = 1$ has the following optimal parameters:

$$\begin{aligned} \alpha &= 0.343 \\ \theta_1 &= 0.4142. \end{aligned}$$

This amounts to a gain of 1.38 dB for $d_{H,2}$ even or large $d_{H,2}$. Decision regions are shown in Fig. 6.

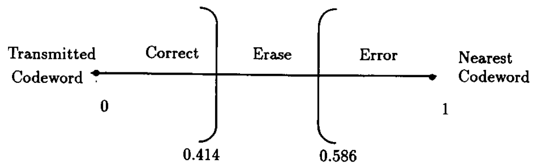

Fig. 6. Single-branch soft-decision decoding with optimal thresholds.

TABLE I
GAIN OVER HARD DECISION DECODING

Number of decoders	Optimal Quantization	Uniform Quantization
1	1.37dB	1.37dB
2	2.24dB	2.15dB
3	2.55dB	2.43dB
4	2.70dB	2.57dB
5	2.78dB	2.66dB
∞	3.01dB	3.01dB

- For $z = 2$, the optimal thresholds and the error-correcting capability are given by

$$\alpha = 0.4187$$
$$\theta_1 = 0.35288$$
$$\theta_2 = 0.45757.$$

This amounts to a gain of 2.24 dB for large $d_{H,2}$.

Asymptotically, as z becomes large, the gain is 3 dB over the baseline (for large $d_{H,2}$). The results given in [4] show that for a large number of decoders operating in parallel the receiver can correct up to half the minimum Euclidean distance of the code. Only four decoders are needed to provide nearly 95% of the maximum Euclidean error-correcting capability of the overall code.

In comparison, we determine the optimal thresholds when the quantizer is constrained to be uniform; that is, the decoder is restricted to thresholds with the constraints $\theta_{k+1} - \theta_k = \theta_k - \theta_{k-1}, k = 2, \cdots, z-1$. The gain (in decibels) of an optimal uniform quantizer over hard-decision outer decoding can be shown to be

$$G = 10 \log_{10} \left(\frac{8z^2}{(2z - 1 + \sqrt{2})^2} \right).$$

Table I summarizes the relative performance of the two decoding algorithms: One with the optimal uniform quantized thresholds and the other with the unconstrained optimal quantized thresholds.

C. Noncoherent Case—Ratio Threshold (M-ary FSK)

Consider a noncoherent channel with M-ary code symbols transmitted over a continuous additive white Gaussian channel, using orthogonal Frequency Shift Keying (FSK). The transmitted signal is of the form

$$s(t) = \sqrt{2P} \cos(\omega_0 t + \phi) p_T(t)$$

where P is the power, T is the symbol duration, ϕ is a phase offset (unknown to the receiver), and ω_0 is the frequency used to transmit signal 0. The jamming signal has a similar form (during the interval $[0, T]$)

$$j(t) = \sum_{i=0}^{M-1} \sqrt{2J_i} \cos(\omega_i t + \psi_i) p_T(t)$$

where J_i is the jamming power in frequency i and ψ_i is the phase. The received signal is

$$r(t) = s(t) + j(t).$$

The received signal is demodulated according to Fig. 7.

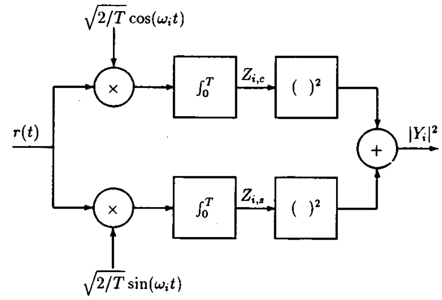

Fig. 7. Filter i for noncoherent demodulation of M-ary FSK.

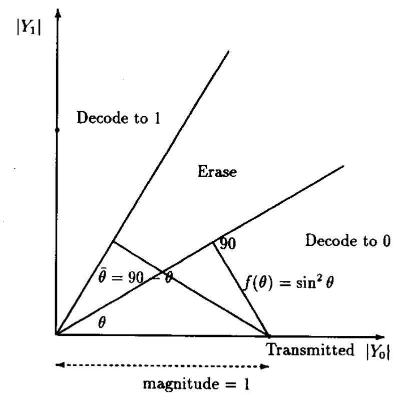

Fig. 8. Noncoherent case: Viterbi ratio threshold.

Assuming the energy per code symbol is E then the model for the situation of interest is

$$Z_{0,c} = \sqrt{E} \cos(\phi) + \sqrt{J_0/T} \cos(\psi)$$
$$Z_{0,s} = \sqrt{E} \sin(\phi) + \sqrt{J_0/T} \sin(\psi)$$

and for $i = 1, \cdots, M-1$

$$Z_{i,c} = \sqrt{J_i/T} \cos(\psi_i)$$
$$Z_{i,s} = \sqrt{J_i/T} \sin(\psi_i).$$

Since we are interested in a worst case scenario we assume the phase of the jamming signal is 180° opposite of the transmitted signal. In addition, the jammer has nonzero noise in only one of the $M - 1$ other frequencies (call it frequency ω_1). With the above assumptions it is not difficult to show that

$$|Y_0| = E\left(1 - \sqrt{\frac{J_0}{P}}\right)$$
$$|Y_1| = E\sqrt{\frac{J_1}{P}}$$
$$|Y_i| = 0, \qquad i = 2, \cdots, M-1$$

where $E_{J_i} = J_i T$ is the energy of the jamming signal at frequency i. Then using ratio thresholding as an erasure criteria, the demodulator puts out symbol 0 if and only if

$$\max_{i=1,\cdots,M-1} \frac{|Y_i|}{|Y_0|} \le \tan\theta.$$

The decision regions (for $M = 2$) are shown in Fig. 8.

For the worst case noise (see Fig. 8), the cost function is the total noise power $(J_0 + J_1)$ necessary to cause a transition to either the erasure region or error region. It is not too hard to show that $f(\theta) = \sin^2(\theta)$. Let $f(\theta_i)$ be defined as follows:

$$f(\theta_i) = \sin^2\theta_i, \quad \theta_i \in \left[0, \frac{\pi}{2}\right]$$
$$f(\overline{\theta}_i) = f\left(\frac{\pi}{2} - \theta_i\right) = \cos^2\theta_i.$$

Thus the decision rule for a decoder with threshold θ is

$$\text{Correct if} \quad \frac{|Y_1|}{|Y_0|} \leq \tan\theta$$
$$\text{Erase if} \quad \tan\theta < \frac{|Y_1|}{|Y_0|} < \cot\theta$$
$$\text{Error if} \quad \frac{|Y_1|}{|Y_0|} \geq \cot\theta.$$

- For the single-branch case ($z = 1$), the optimal threshold can be found to be

$$\theta = \tan^{-1}\frac{1}{\sqrt{2}} = 35.3^\circ$$

and the error-correcting capability is $\alpha = 0.667$. This results in an improvement of 1.25 dB for large $d_{H,2}$.
- $z = 2$ results in the following equations:

$$f(\overline{\theta}_1) + f(\theta_0) = \alpha$$
$$f(\overline{\theta}_2) + f(\theta_1) = \alpha$$
$$2f(\theta_2) = \alpha.$$

This gives the following equations:

$$\cos^2\theta_1 = \alpha$$
$$\cos^2\theta_2 + \sin^2\theta_1 = \alpha$$
$$2\sin^2\theta_2 = \alpha.$$

Solving these equations gives

$$\theta_1 = 26.5^\circ$$
$$\theta_2 = 39.23^\circ.$$

The improvement is 2.04 dB (for large $d_{H,2}$).

For an arbitrary number of branches the equations are

$$1 - \sin^2\theta_1 = \alpha$$
$$1 - \sin^2\theta_2 + \sin^2\theta_1 = \alpha$$
$$1 - \sin^2\theta_3 + \sin^2\theta_2 = \alpha$$
$$1 - \sin^2\theta_z + \sin^2\theta_{z-1} = \alpha$$
$$\sin^2\theta_z = \alpha/2.$$

It is easy to see that if we sum all these equations we eliminate the variables θ_k and are left with $z = \alpha(z + 1/2)$. Thus

$$\alpha = \frac{2z}{2z+1}.$$

We note here that α can be thought of as the error-correcting capability of the decoding algorithm. Once α is determined we can determine the values for θ quite easily.

$$\sin^2\theta_i = \frac{i}{2z+1}, \qquad i = 1, 2, \cdots, z.$$

The coding gain is given by 2α (for large $d_{H,2}$) and approaches 3 dB as z becomes large.

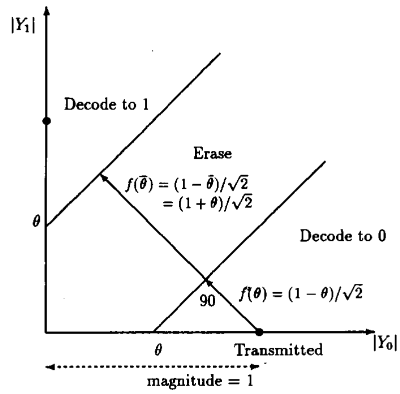

Fig. 9. Decision regions for difference metric.

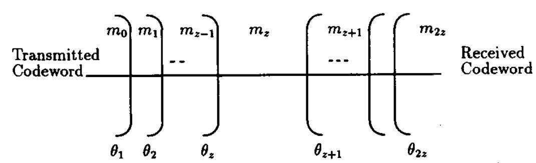

Fig. 10. Decoding with arbitrary metrics.

D. Noncoherent Case—Difference Threshold (M-ary FSK)

In this case, the demodulator forms the difference between the largest energy and the next largest energy, and compares the difference to a threshold similar to the ratio threshold case. According to Fig. 9, the cost function is the squared noise length necessary to cause a transition to either the erasure region or error region. It is not too hard to show that if the jammer has phase opposite to the signal of interest, the minimum squared noise length to cause a transition is $f(\theta) = (1-\theta)^2/2$. This is essentially the same as Euclidean distance and a simple transformation can be applied to determine the solution of this game from the solution to the Euclidean distance problem.

The decision rule for a decoder with threshold θ is

$$\text{Decide 0 if} \quad |Y_1| < |Y_0| - \theta$$
$$\text{Erase if} \quad |Y_0| - \theta \leq |Y_1| \leq |Y_0| + \theta$$
$$\text{Decide 1 if} \quad |Y_1| > |Y_0| + \theta.$$

This is equivalent to the Euclidean distance formulation. The gain over baseline decoding is the same. Namely for $z = 1$ we get 1.38-dB gain and for $z = 2$ we get 2.24 dB. Asymptotically, the gain is 3 dB. Note that difference thresholding is about 0.2 dB better (for $z = 2$) than ratio thresholding but requires knowledge of the received-signal amplitude whereas ratio thresholding does not.

III. Correlation Decoding

We formulate the problem of finding soft-decision decoding algorithms with arbitrary metrics that achieve maximum error-correcting capability. The game of interest is that of finding optimal decoding metrics and thresholds. Consider quantizing the demodulator output to one of z levels around each of the possible transmitted symbols. We need to assign a metric to each quantization region to use in the Viterbi decoder; moreover, the z quantization regions are determined by a fixed set of thresholds $\boldsymbol{\theta} = \{\theta_1, \theta_2, \cdots, \theta_z\}$. Let $m_i, 0 \leq i \leq z$ be the metric assigned to region i (as shown in Fig. 10).

The jammer's cost function is specified by $f(\theta_i)$. Consider two codewords separated by the minimum distance of the code (for convolutional codes this would be two paths that diverge and remerge, e.g., a minimum distance path). Let $\nu_i \in [0,1]$ be the number of demodulated symbols along the path that are quantized to region i, normalized by the path minimum distance d. The following game is then of interest:

$$\max_{\boldsymbol{\theta}, \boldsymbol{m}} \min_{\boldsymbol{v}} \sum_{i=0}^{2z} \nu_i f(\theta_i) \tag{2}$$

where $\boldsymbol{m} = \{m_0, m_1, m_2, \cdots, m_{2z}\}$, and $\boldsymbol{v} = \{\nu_0, \nu_1, \cdots, \nu_{2z}\}$. The minimization is conditioned on a decoder error; that is, such that

$$\sum_{i=0}^{2z} \nu_i m_i < \sum_{i=0}^{2z} \nu_i m_{2z-i} \tag{3}$$

$$\sum_{i=0}^{2z} \nu_i = 1 \tag{4}$$

$$m_0 \geq m_1 \geq m_2 \cdots \geq m_{2z} \tag{5}$$

$$f(\theta_0) \leq f(\theta_1) \leq f(\theta_2) \leq \cdots \leq f(\theta_{2z}). \tag{6}$$

It is assumed that $\theta_0 = 0,\ f(0) = 0$, and

$$\theta_{z+i} = 1 - \theta_{z-i+1}, \theta_1 \leq \theta_2 \leq \cdots \leq \theta_z.$$

Obviously, the absolute values for the metrics do not matter, since scaling $\boldsymbol{m}$ by a constant does not change the decoder.

At this point, it is intuitive that asymptotically with z, the error-correcting capability of the maximum-likelihood decoder described above is smaller than or equal to that of the parallel decoder described in the previous sections. Next, the error-correcting capability of the correlation decoder is evaluated for finite z.

Assume that d is large enough so that ν_i can be assumed real-valued. It will soon be shown that this assumption can be relaxed. Then the linear programming (LP) problem

$$\min_{\boldsymbol{v}} \sum_{i=0}^{2z} \nu_i f(\theta_i) \tag{7}$$

with the linear constraints given by (3) and (4) has a solution at one particular extreme point (or vertex) of the feasible region. The extreme points are the $2z+1$-dimensional vectors $\boldsymbol{v}$ with $2z-1$ components having zero values (the simplex theorem [6]). The nonzero extreme points are the solution of the equations

$$\nu_l + \nu_k = 1 \tag{8}$$

$$(m_{2z-l} - m_l)\nu_l + (m_{2z-k} - m_k)\nu_k = 0 \tag{9}$$

where $k = 0, 1, \cdots, 2z+1$ and $l < k$. Let

$$\Delta_k \triangleq m_{2z-k} - m_k$$

then

$$\nu_k = \frac{\Delta_l}{\Delta_l - \Delta_k} \tag{10}$$

$$\nu_l = -\frac{\Delta_k}{\Delta_l - \Delta_k}. \tag{11}$$

Then, the solution to the LP problem is

$$\min_{k,l} \frac{f(\theta_k)\Delta_l - f(\theta_l)\Delta_k}{\Delta_l - \Delta_k} \tag{12}$$

and the game is finding the solution for

$$\max_{\Delta, \boldsymbol{\theta}} \min_{k,l} \frac{f(\theta_k)\Delta_l - f(\theta_l)\Delta_k}{\Delta_l - \Delta_k} \tag{13}$$

where $\Delta = \{\Delta_0, \cdots, \Delta_z\}$. Notice that $\Delta_0 \leq \Delta_1 \leq \cdots \leq \Delta_z$.

The solution can be shown to be an equalizer strategy such that, for some constant α_z

$$\frac{f(\theta_k)\Delta_l - f(\theta_l)\Delta_k}{\Delta_l - \Delta_k} = \alpha_z, \quad \text{for } k = 1, \cdots, 2z, l < k \tag{14}$$

where

$$\Delta_z = 0, \Delta_{z+i} = -\Delta_{z-i}, i = 1, 2, \cdots, z.$$

Thus setting $k = z+i$ and $l = z-i$ for $i = 1, \cdots, z$ and setting $l = 0, k = z$ in (14) yields the same $z+1$ equations (1) in $z+1$ unknowns $\alpha, \theta_1, \cdots, \theta_z$ with α replaced by 2α. Thus for optimal metrics, the error-correcting capability is the same as achieved in the parallel decoders described earlier, and the optimal thresholds are the same. Recall that the error-correcting capability for the decoders in Section II is $d_{H,2}\alpha/2$, and the error-correcting capability for the decoder analyzed in this section is $d_{H,2}\alpha$. The rest of the equations are constraints on Δ_i. Next it is shown that the assumption ν_i is real-valued is not needed.

With optimal choices for $\boldsymbol{\theta}$ and $\boldsymbol{\Delta}$, the equalizer strategy (14) indicates that a jammer can place the d symbols according to (10) and (11) for all values of *valid* i and k. That is, the jammer strategy that causes the minimum cost *not* correctable by the decoding algorithm is not unique. For arbitrary $d_{H,2}$, letting $k = z$ in (10) results in $\nu_z = 1, \nu_i = 0, i \neq z$; thus the assumption ν_i is real-valued is not needed. Moreover, if d is even, then $k = z+i$ and $l = z-i, i = 1, 2, \cdots, z$ result in the following valid optimal strategies:

$$\nu_{z+i} = \tfrac{1}{2}$$

$$\nu_{z-i} = \tfrac{1}{2}.$$

Combining two of the equalizer equations

$$f(\theta_z) = \alpha$$

$$f(\theta_{z+i}) + f(\theta_{z-i}) = 2\alpha$$

gives

$$f(\theta_z) = \frac{f(\theta_{z+i}) + f(\theta_{z-i})}{2}.$$

Therefore, if $d_{H,2}$ is odd, say $d_{H,2} = 2k+1$, the jammer can place k symbols in region $z+i$, k symbols in region $k-i$, and one symbol in region z (which according to the last equation is equivalent to invalid placing $k+1/2$ symbols in each region $k-i$ and $k+i$). Thus

$$\nu_{z+i} = \frac{k}{d_{H,2}}$$

$$\nu_{z-i} = \frac{k}{d_{H,2}}$$

$$\nu_z = \frac{1}{d_{H,2}}.$$

As mentioned previously, other strategies which satisfy equations (2) for the jammer exist, but are not of interest because they require the jammer to have a fractional—even irrational—number of components at a certain length.

The following example will clarify the strategies analyzed in this section for the additive channel case with coherent reception; that is, $f(\theta) = \theta^2$.

- $z = 1$. As expected, for the single-branch case it is easy to show that there are no restrictions on Δ_0.
- The equations for $z = 2$ are

$$f(1-\theta_1) = 2\alpha$$

$$f(1-\theta_2) + f(\theta_1) = 2\alpha$$

$$f(\theta_2) = \alpha.$$

These equations can be solved for θ_1, θ_2, and α. The results are the same as the two-branch parallel decoder. $\theta_1 = 0.3529$, $\theta_2 = 0.4576$, and $\alpha = 0.2094$. The equations for $k = 1, l = 0$ yield the solution for Δ_0 and Δ_1.

$$\frac{\Delta_1}{\Delta_0} = 1 - \left(\frac{\theta_1}{\theta_2}\right)^2 = 0.40519.$$

Using the results for (10) and (11) we obtain the following possible minimum-cost jamming strategies which cause an error:

$$\begin{aligned} \nu_2 &= 1 \\ \nu_0 &= 1/2 \quad \nu_4 = 1/2 \\ \nu_1 &= 1/2 \quad \nu_3 = 1/2. \end{aligned}$$

The other equations yield solutions for ν_i which are not in the interval $[0, 1]$ and thus are not valid or equivalent.

We note here that the strategies for noise vectors that are split evenly between two quantization intervals cannot be implemented with two codewords of odd distance. Nevertheless, another jamming strategy that has the same cost can be implemented. For example, consider the strategy with $\nu_1 = 1/2, \nu_3 = 1/2$. If the distance is even this strategy causes no problems as half of the positions in which the two codewords differ are placed in region 3 by the jammer and half are placed in region 1. If the distance is odd (say $d_{H,2} = 2k + 1$) then the same cost (and metric difference) is achieved by the strategy $\nu_1 = k/d_{H,2}, \nu_2 = 1/d_{H,2}, \nu_3 = k/d_{H,2}$. That the cost is the same is evident by examining the equations above. Similar conclusions can be made for other strategies shown above. Thus the strategies that are also valid are (for $d = 2k + 1$)

$$\begin{aligned} \nu_0 &= k/d_{H,2} \qquad \nu_2 = 1/d_{H,2} \qquad \nu_4 = k/d_{H,2} \\ \nu_1 &= k/d_{H,2} \qquad \nu_2 = 1/d_{H,2} \qquad \nu_3 = k/d_{H,2}. \end{aligned}$$

Notice that an optimal strategy such as

$$\nu_0 = \frac{1}{1 + \Delta_1/\Delta_0} \qquad \nu_3 = 1 - \nu_0$$

is not valid, since $(\theta_1/\theta_2)^2$ is irrational.

The error-correcting capability for the optimal jammer strategies above is 0.2094.

- $z = 3$ gives for $k = 1, l = 0$ and $k = 2, l = 0$

$$\begin{aligned} \frac{\Delta_1}{\Delta_0} &= 1 - \left(\frac{\theta_1}{\theta_3}\right)^2 \\ &= 0.51710 \\ \frac{\Delta_2}{\Delta_0} &= 1 - \left(\frac{\theta_2}{\theta_3}\right)^2 \\ &= 0.23014. \end{aligned}$$

The error-correcting and optimum thresholds capability are rewritten below.

$$\alpha = 0.2250 \qquad \theta_1 = 0.3294 \qquad \theta_2 = 0.4160 \qquad \theta_3 = 0.4741$$

Other values for k and l will result in the same relations.

For $d_{H,2}$ even, the following possible minimum-cost jamming strategies will cause an error:

$$\begin{aligned} \nu_3 &= 1 \\ \nu_0 &= 1/2, \quad \nu_6 = 1/2 \\ \nu_1 &= 1/2, \quad \nu_5 = 1/2 \\ \nu_2 &= 1/2, \quad \nu_4 = 1/2. \end{aligned}$$

For $d_{H,2} = 2k + 1$, the following jammer strategies are optimum and valid:

$$\begin{aligned} \nu_3 &= 1 \\ \nu_0 &= k/d_{H,2}, \quad \nu_3 = 1/d_{H,2}, \quad \nu_6 = k/d_{H,2} \\ \nu_1 &= k/d_{H,2}, \quad \nu_3 = 1/d_{H,2}, \quad \nu_5 = k/d_{H,2} \\ \nu_2 &= k/d_{H,2}, \quad \nu_3 = 1/d_{H,2}, \quad \nu_4 = k/d_{H,2}. \end{aligned}$$

The other equations yield solutions for ν_i which are not in the interval $[0, 1]$ and thus are not valid or equivalent. Again, the error-correcting capability for the above strategies is 0.2250.

IV. Conclusions

We have described several classes of decoding algorithms that have optimal error-correcting capability for a given channel model. The first class is bounded-distance decoding, and is characterized by a set of thresholds to determine erasure regions in a parallel decoding arrangement. The second class is maximum-likelihood decoding characterized by a set of thresholds that determine quantization regions and metrics assigned to these quantization regions. It was shown that for the same number of thresholds and certain constraints on the metrics, the two classes have the same error-correcting capability. Intuitively, this could be true asymptotically with the number of thresholds, and it is not obvious that it would be the same for a finite number of thresholds.

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