Optimal Selection of Reed–Solomon Code Rate and the Number of Frequency Slots in Asynchronous FHSS–MA Networks

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Abstract— **In this paper we consider the performance of Reed–Solomon codes in asynchronous FHSS–MA networks. When q denotes the number of frequency slots available to the network and r denotes the rate of the Reed–Solomon code, we obtain optimal (q, r) pairs that achieve a given performance criteria with minimum bandwidth expansion (q/r) for a given number of active users. We show that the optimal code rate rapidly converges to a constant value and the optimum number of slots increase approximately linearly as the number of active users increase. This suggests that one should fix the code rate and increase the number of slots to accommodate the increasing number of users in the network under a given performance criteria with minimum bandwidth expansion.**

I. Introduction

IN [1], Kim and Stark considered the performance of Reed–Solomon (RS) codes in frequency-hop spread-spectrum multiple-access (FHSS–MA) networks. The performance measures considered in [1] were the normalized throughput and achievable regions for the RS code rate that guarantee a given performance criteria. The normalized throughput w defined as rKP_c/q (r denotes the rate of the RS code, K denotes the number of active users in the network, P_c denotes the probability that a RS codeword will be transmitted through the network error free and q denotes the number of frequency slots available) takes into account the bandwidth expansion caused by the code rate r which is less than unity and the number of frequency slots q which is greater than unity. But the analysis of the achievable region does not take this fact into account. That is, for a given K, q and n (the length of the RS code), the maximum code rate r was found that guarantees the RS codeword error probability P_e to be less than a prespecified value of $\hat{P}_e$.

In this paper, instead of fixing q, we find optimal (q, r) pairs that not only achieves $P_e < \hat{P}_e$ but also does this with minimum bandwidth expansion z defined by

$$z = \frac{q}{r}. \tag{1}$$

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As in [1], we consider the cases when perfect side information is used to erase the hops that were hit and when hard decisions are made without perfect side information. We consider the case when practical finite length RS codes are used and the asymptotic case when RS codes with very large lengths are used.

II. System Model and Performance Measures

The system model employed in this paper is the same as the one used in [1]. We consider a FHSS–MA channel with K active users and q (frequency) slots. To each user is assigned a memoryless hopping sequence that is also independent from user to user. We consider asynchronous hopping. In this case the probability that a frequency slot is shared with at least one other user denoted by $P_{h,K}$, is given by [2]

$$P_{h,K} = 1 - (1 - p_h)^{K-1} \tag{2}$$

where

$$p_h = \frac{1}{q}\left(2 - \frac{1}{q}\right) \tag{3}$$

is the probability that a slot is shared with one other user in the network. We assume that the network is homogeneous, i.e., that all the transmitter-receiver pairs are identical. The data source of each user is encoded with an (n, k) Reed–Solomon encoder where n denotes the length of the code and the rate of the code is given by $r = k/n$. Each hop is assumed to contain one RS code symbol and one packet is defined to be one RS codeword. Furthermore, the transmission of packets among the users are assumed to be synchronized. For the sake of simplicity we assume that the majority of the errors are caused by multiple-access interference and the background noise is negligible.

Two receiver models are considered. First, we consider a receiver that has the ability to determine whether or not a hop was hit (i.e., that a slot was shared with at least one other user). This type of information is usually referred to as perfect side information. For this case, the receiver erases the symbols transmitted in those hops that were hit. Hence for this receiver model, the resulting discrete channel can be modeled as an M-ary symmetric erasure channel with erasure probability equal to $P_{h,K}$ where M is the alphabet size of the RS code. Next, we consider a receiver that does not have this ability and simply makes hard decisions based on the noncoherent demodulator outputs. In this case, the error model employed is that whenever a hop is hit, the demodulator output is equally likely to be any one of the M-ary symbols. Hence for this receiver model, the resulting discrete channel is modeled as an M-ary symmetric errors channel with error probability equal to $P_{e,K} = ((M-1)/M)P_{h,K}$.

The performance measure we consider is the achievable region which consists of vectors $(q(K), r(K))$ that guarantees the RS codeword error probability P_e to be less than some specified value $\hat{P}_e$ for a given K. In particular, we are interested in the vectors $(q_{\mathrm{opt}}(K), r_{\mathrm{opt}}(K))$ in the achievable region that minimize the bandwidth expansion factor z defined in (1). We refer to this minimum bandwidth expansion factor as $z_{\mathrm{opt}}(K)$. The vectors $(q_{\mathrm{opt}}(K), r_{\mathrm{opt}}(K))$ are the minimum bandwidth (q, r) pairs that ensure $P_e < \hat{P}_e$ for RS codes and a given number of active users.

III. Perfect Side Information Available

We start the analysis with the first receiver model described in the previous section where the receiver erases the symbols transmitted in the hops that were hit. It is well known that when an (n, k) RS code is employed over an M-ary symmetric erasure channel, the codeword error probability is given by [3]

$$P_e(K,n,k,q) = \sum_{i=n-k+1}^{n} \binom{n}{i} P_{h,K}^{i}(1-P_{h,K})^{n-1}. \quad (4)$$

A. Finite Length Codewords

First we consider the case where finite length RS codes are used and $P_e(K,n,k,q) < \hat{P}_e$, is chosen as the performance criteria that the system must satisfy. In this case, we may find $z_{\mathrm{opt}}(K)$ and $(q_{\mathrm{opt}}(K), r_{\mathrm{opt}}(K))$ through a computer search over all possible (q, r) pairs that satisfy $P_e(K,n,k,q) < \hat{P}_e$. Another method is to use the Gaussian approximation for $P_e(K,n,k,q)$ developed in [1] which is given by

$$P_e(K,n,k,q) \approx 1 - \Phi\left(\frac{n-k-nP_{h,K}}{\sqrt{nP_{h,K}(1-P_{h,K})}}\right) \quad (5)$$

where $\Phi(x) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt$. It is also shown in [1] that the condition $P_e(K,n,k,q) < \hat{P}_e$ may be translated to a bound on r as follows:

$$r < 1 - P_{h,K} - \alpha\sqrt{P_{h,K}(1-P_{h,K})/n} \quad (6)$$

where $\alpha = \Phi^{-1}(1-\hat{P}_e)$. Hence, under this approximation,

$$z_{\mathrm{opt}}(K) = \min_{q\in\boldsymbol{Q}} \frac{q}{r(K,q)} \quad (7)$$

where

$$r(K,q) = 1 - P_{h,K} - \alpha\sqrt{P_{h,K}(1-P_{h,K})/n} \quad (8)$$

and $\boldsymbol{Q}$ is the set of q that ensures $0 < r(K,q) \le 1$. The vectors $(q_{\mathrm{opt}}(K), r_{\mathrm{opt}}(K))$ are given by

$$q_{\mathrm{opt}}(K) = \operatorname*{argmin}_{q\in\boldsymbol{Q}} \frac{q}{r(K,q)} \quad (9)$$

$$r_{\mathrm{opt}}(K) = r(K, q_{\mathrm{opt}}(K)). \quad (10)$$

B. Asymptotic Analysis

Here, we consider the case when the length of the RS codeword n tends to infinity while the code rate $r = k/n$ is held constant. In this case it was shown in [1] that error-free communications is possible if $r < (1-p_h)^{K-1}$. That is, if

$$r < \left(1 - \frac{1}{q}\left(2 - \frac{1}{q}\right)\right)^{K-1}. \quad (11)$$

It is easy to show that

$$z_{\mathrm{opt}}(K) = \min_{\{q\ge 2,\, r<(1-p_h)^{K-1}\}} \left(\frac{q}{r}\right) \quad (12)$$

is given by

$$z_{\mathrm{opt}}(K) = \frac{q_{\mathrm{opt}}(K)}{r_{\mathrm{opt}}(K)} \quad (13)$$

where

$$r_{\mathrm{opt}}(K) = \left[1 - \frac{1}{(2K-1)}\right]^{2(K-1)} \quad (14)$$

$$q_{\mathrm{opt}}(K) = 2K - 1. \quad (15)$$

For this case, we note that

$$\lim_{K\to\infty} r_{\mathrm{opt}}(K) = e^{-1}. \quad (16)$$

Hence, as K becomes large, the optimal rate with the optimum number of frequency slots converges to e^{-1}. It will be shown in Section V that this convergence is quite fast.

IV. No Side Information

Now, we consider the second receiver model described in Section II where the receiver simply makes hard decisions. In this case the codeword error probability is given by [3]

$$P_e(K,n,k,q) = \sum_{i=\frac{n-k}{2}+1}^{n} \binom{n}{i} P_{e,K}^{i}(1-P_{e,K})^{n-i} \quad (17)$$

where $n-k$ is assumed to be even for simplicity.

A. Finite Length Codewords

As before, for this case, we may employ a computer search to find $z_{\mathrm{opt}}(K)$ and $(q_{\mathrm{opt}}(K), r_{\mathrm{opt}}(K))$ or we may resort to the Gaussian approximation. Using the Gaussian approximation for P_e developed in [1], we can show that

$$z_{\mathrm{opt}}(K) = \min_{q\in\boldsymbol{Q}} \frac{q}{r(K,q)} \quad (18)$$

where

$$r(K,q) = 1 - 2P_{e,K} - 2\alpha\sqrt{P_{e,K}(1-P_{e,K})/n} \quad (19)$$

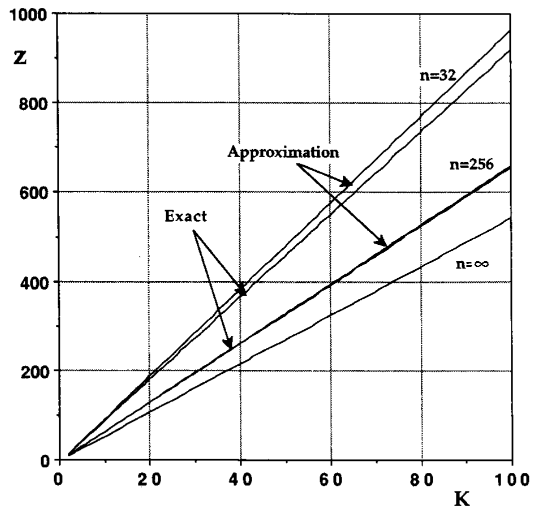

Fig. 1. z_{opt} versus K, perfect side information.

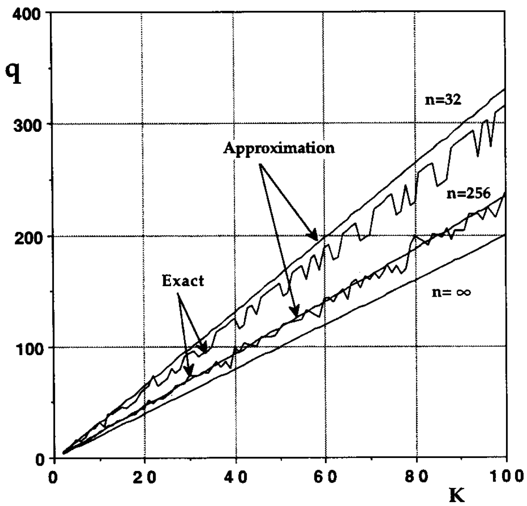

Fig. 2. q_{opt} versus K, perfect side information.

and $\boldsymbol{Q}$ is the set of q that ensures that $0 < r(K, q) \leq 1$ and $\alpha = \Phi^{-1}(1 - \hat{P}_e)$. The vectors $(q_{\text{opt}}(K), r_{\text{opt}}(K))$ are given by

$$q_{\text{opt}}(K) = \operatorname*{argmin}_{q \in \boldsymbol{Q}} \frac{q}{r(K, q)} \tag{20}$$

$$r_{\text{opt}}(K) = r(K, q_{\text{opt}}(K)) \tag{21}$$

B. Asymptotic Analysis

Carrying out similar computations as in Section III, we have that

$$z_{\text{opt}}(K) = \min_{q > q_{\min}} \frac{q}{r(K, q)} \tag{22}$$

where

$$r(K, q) = 2\left(1 - \frac{1}{q}\left(2 - \frac{1}{q}\right)\right)^{K-1} - 1 \tag{23}$$

and

$$q_{\min} = \frac{1}{1 - \left(\frac{1}{2}\right)^{\frac{1}{2(K-1)}}}. \tag{24}$$

The condition $q > q_{\min}$ is needed to ensure $0 < r \leq 1$. The vector $(q_{\text{opt}}(K), r_{\text{opt}}(K))$ that achieves this is given by

$$q_{\text{opt}}(K) = \operatorname*{argmin}_{q > q_{\min}} \frac{q}{r(K, q)} \tag{25}$$

$$r_{\text{opt}}(K) = r(K, q_{\text{opt}}). \tag{26}$$

For large K $q_{\text{opt}}(K) \to 6.35K$ and $r \to 0.460$.

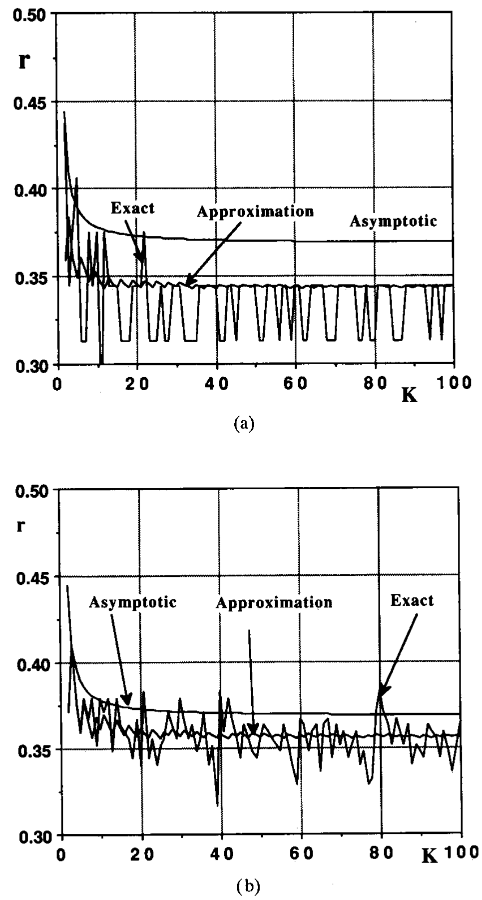

Fig. 3. r_{opt} versus K, perfect side information: (a) $n = 32$, (b) $n = 256$.

V. Numerical Results

In this section, we present the numerical results. When RS codes with finite length codewords are used, the minimum performance criteria was taken to be $P_e < 10^{-2}$. In Figs. 1–3, we plot z_{opt}, q_{opt}, and r_{opt} for the case when perfect side information is used to erase the hops that were hit. First

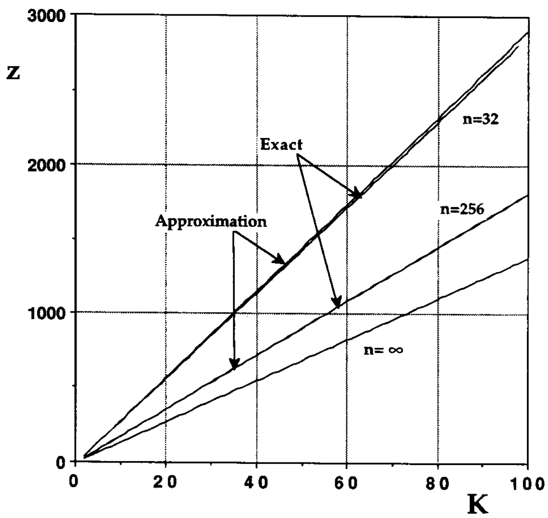

Fig. 4. z_{opt} versus K, no side information.

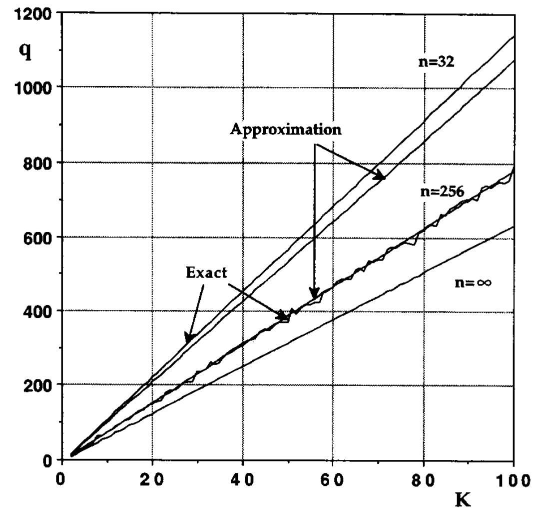

Fig. 5. q_{opt} versus K, no side information.

notice that the approximation to the exact value for the optimal bandwidth expansion (q/r) is extremely accurate for $n = 256$ and fairly accurate for $n = 32$. The approximation and the exact value for the optimal number of frequency slots shown in Fig. 2. are also very close. The jaggedness of the exact curves are due to the fact that we are optimizing over a set of integer number of frequency slots q and code dimensions k in the exact calculation, while the approximation treats these as continuous variables. The optimal rate as a function of the number of users is shown in Fig. 3(a) and (b) for $n = 32$ and $n = 256$. Besides the fluctuation due to integer parameter values the optimal code rate rapidly converges to the asymptotic result for large K. The large K results appear to be valid for K larger than around 20. Also since the optimal number of slots increase linearly in K, we see from Fig. 1 that the minimum bandwidth expansion factor increases linearly in K for large K. This shows that the increasing number of users in the network should be accommodated by increasing q and fixing the code rate to ensure minimum bandwidth expansion. The asymptotic optimal bandwidth expansion with side information is then $z \approx 2e \approx 5.43K$.

We note here that the minimum bandwidth expansion factor required for the asymptotic case when the number of slots q is fixed and the bandwidth expansion factor is minimized only over the code rate shows that without an appropriate choice of the vector (q, r), a FHSS–MA network could use unnecessarily large amount of bandwidth to achieve a given performance criteria.

In Figs. 4–6, we show plots similar to those shown in Figs. 1–3 for the case when no side information is available and simple hard decisions are made. Similar arguments made for the perfect side information case applies to this case with little modification. For the case when very long RS codewords are employed, the asymptotic (as $K \to \infty$) value of r_{opt} tends to 0.460 and the optimal number of frequency slots approaches $6.35K$ resulting in bandwidth expansion of $13.82K$ or a loss of a factor of 2.54 relative to a system with side information.

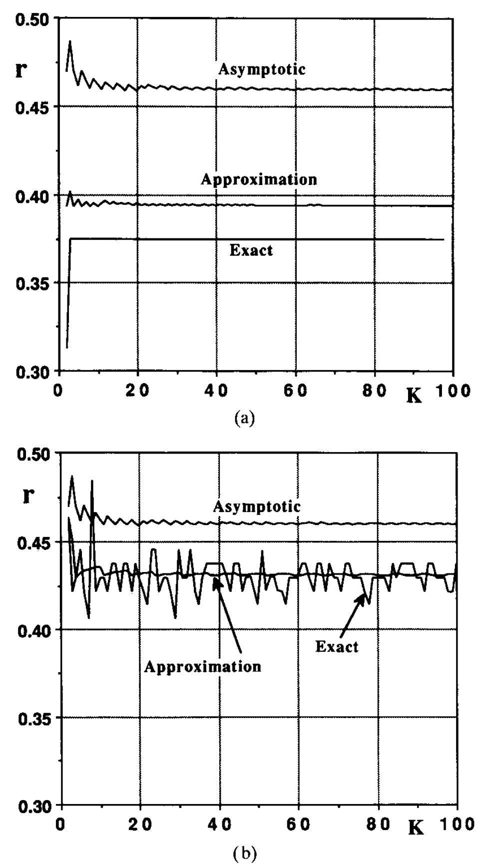

Fig. 6. r_{opt} versus K, no side information: (a) $n = 32$, (b) $n = 256$.

VI. Conclusion

In this paper, we derived the optimal number of slots and the optimal code rate that should be employed to accommodate a certain number of users under a minimum performance criteria in an asynchronous FHSS–MA network with Reed–Solomon coding. We find that large reductions in the required bandwidths are possible with optimal choices of q and r. Also, the results showed that the increasing load on the network (increasing K) should be accommodated by fixing the code rate and increasing the number of slots available to the network. It would be of interest to investigate optimal spreading/coding for other models for the interference and other coding schemes.

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