Probability of Error in Frequency-Hop Spread-Spectrum Multiple-Access Communication Systems with Noncoherent Reception

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***Abstract*—In this paper we develop expressions for the probability of error for asynchronous frequency-hop spread-spectrum multiple-access networks using Markov hopping patterns and binary frequency shift keying with one symbol transmitted per hop. The expressions are exact when there is one interfering user and orthogonal BFSK employed. They provide excellent approximations when there are more than one interfering user. The model employed takes into account the random phases, random data bits and accounts for the effects due to random delays of the interfering signals. The model also incorporates a finite number of different power levels in the network at the intended receiver. It is also shown that the error probability when Markov hopping patterns are employed is a good approximation to the error probability when memoryless hopping patterns are employed. Finally, by computing the channel capacity and the associated throughput, a simple hard decisions receiver is shown to perform much better than a receiver using perfect side-information to erase the symbols transmitted on hops that were hit when all the users have same power and one binary symbol is transmitted per hop.**

I. Introduction

In this paper we consider an asynchronous frequency-hop spread-spectrum multiple-access (AFHSS–MA) network employing binary frequency shift keying (BFSK) to transmit one binary symbol per hop. Two approximations usually made in analyzing this type of systems are that the hits due to multiple-access interference are independent from hop to hop (the independence assumption) and that the error probability is upper bounded by $1/2$ whenever a hop is hit by multiple-access interference [1]–[3] (the $1/2$-bound). Recently, it has been shown by Hegde and Stark [4] that though the hits exhibit an underlying Markov structure, the independence assumption is quite accurate. On the other hand, there has been no definitive solution to computing the probability of error when a hop is hit by multiple-access interference in an AFHSS–MA network.

In [5], a Gaussian approximation was used to approximate the probability of error. In [6], an exact expression for the probability of error was derived for orthogonal BFSK when a hop is hit by one interfering user, and Monte–Carlo simulations were performed to evaluate the error performance for the cases when there are a small number of interfering users (≤ 4). On the other hand in [7], the effect of nonorthogonality of the interfering signals due to the asynchronocity (i.e., random delays) of different users was neglected in the analysis by assuming that the frequency separation between the orthogonal BFSK signals are much larger than the minimum required to guarantee orthogonality (we will refer to this as the Geraniotis' assumption), and expressions for the error probability was derived based on this assumption. But in practice, it would be advantageous to use the minimum frequency separation required because for a fixed bandwidth, this would allow for a larger number of frequency slots and thus decrease the probability that a hop is hit by other interfering users. When minimum frequency separation is employed, an interfering signal arriving at the receiver with a nonzero delay will affect the outputs of both of the correlators even if orthogonal BFSK is employed and thus Geraniotis' assumption would no longer be valid. In [8], a Poisson model was used for the number of interfering users and error probability analysis was done under similar assumptions as in [7]. Very recently, Short and Rushforth [9], obtained an exact[1] expression for the error probability for AFHSS–MA using BFSK *conditioned* on the phases, the delays, the power levels, and the data bits of the interfering signals. This paper also incorporated the case when the FSK signals used are nonorthogonal. In most practical AFHSS–MA systems, it is natural to assume that the phases of the signals of different users as seen by a receiver are random variables statistically independent of each other and to all other random variables and are uniformly distributed on $[0, 2\pi)$. Also, when synchronization at the hop level is not achievable (asynchronous hopping), it is reasonable to assume that the delays of the interfering signals as seen by a receiver are also random variables that are independent of all other random variables and uniformly distributed on $(-T, T)$ where T is the duration of a hop. It is theoretically possible to average

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[1] The only assumption made here is that the interference from other frequency-hop slots are negligible which is also the assumption made here.

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the expression given in [9] over the phases, delays and the data bits of the interfering signals, but with K' interfering users, this would mean the evaluation of a $2K'$-fold integral and a K'-fold summation of the expression for the probability of error, which is not very attractive.

In this paper an AFHSS–MA system where each user employs Markov hopping patterns that are independent of each other is considered. Hence, given that a frequency slot is used for the nth hop of a user, the slot used for the $(n+1)$st hop of that user will be uniformly distributed among the available frequency slots other than the one used on the nth hop. It is assumed that the initial phases of the signals of the users are random, independent of each other and to any other random variable, and are uniformly distributed on $[0, 2\pi)$. Then assuming (for $K' \geq 2$) that the outputs of the two correlators are independent, we apply the results from [10], [11] to obtain an approximation for the probability of error (the results are exact for one interfering user). By assuming that the delays and the data bits of the users are independent of each other and are independent from user to user, and from all other random variables, a reasonably compact expression is obtained for the average probability of error given the power levels of the interfering users. The computational complexity of numerically evaluating this expression is dominated by computation of two expectations which are independent of K' (the number of interfering users in a given hop) and the signal-to-noise ratio. Thus once we have computed these two expectations, computing the error probability given a hop is hit by K' users for any signal-to-noise ratio is a simple task.

In the numerical results section, we compare our results to Monte-Carlo simulation results. It is shown that the approximation (assuming the outputs of the correlators are independent) provides an excellent fit to the actual error probability. It is shown that assuming Markov hopping patterns not only greatly simplifies the analysis and gives results that are easily computed, but also gives very good approximations to the probability of error when memoryless hopping patterns are employed. Channel capacity and the associated throughput based on our results for the probability of error are computed to reveal, quite surprisingly, that a simple hard decisions receiver performs better than a receiver using perfect side-information to erase the code symbols transmitted on a hop that was hit. Performance of synchronous FHSS–MA networks is also considered as a special case of asynchronous FHSS–MA networks where the delays of all the interfering users are zero. For this case the expressions for the probability of error given here are exact for all K' given that orthogonal BFSK is employed. It is shown that asynchronous systems have superior performance over synchronous systems. These observations are true for systems with one binary symbol transmitted per hop but may not be true for systems with multiple symbols transmitted per hop (a slow FH system).

In Section II, we give a brief description of our system model and present the analyses that leads to the expression for the probability of error. In Section III, it is shown that our results given excellent approximations to Monte–Carlo simulation results. Also, numerical results are given for the error probability, channel capacity and throughput and in Section IV, we draw conclusions.

II. System Description and Analysis

Before carrying out the analysis we define some notation that will be used throughout this paper. Let K be the number of active users in the network. Let N be the number of power level groups, that is, there are N different power levels as seen by a receiver, and let $\overline{K}_1, \cdots, \overline{K}_N$ be the number of interfering users in each power level group with $\sum_{i=1}^{N} \overline{K}_i = K-1$. Let $\boldsymbol{K} = (\overline{k}_{1,\ldots,}\overline{k}_N)$ be the interference pattern vector with each component $\overline{k}_i$ representing the number of users from each power level group that hit the hop under consideration, and let $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_{K'+1})$ be the amplitude vector with the ith component being the received amplitude of the signal of the ith interfering user. Clearly, $\sum_{i=1}^{N} \overline{k}_i = K'$ is the total number of interfering users that hit the hop. Also let p_h denote the probability that a hop is hit when $K=2$ and q be the number of frequency slots available to the network.

We assume Markov hopping patterns are employed so that the probability of a single interfering user overlapping a given hop for its entire duration is zero. The probability of error when Markov hopping patterns are employed will be shown to be a good approximation to the probability of error when memoryless hopping patterns are employed. This is because the probability of a memoryless hopping pattern generating hops at two consecutive frequency slots is very small for practical values of q, namely, $1/q^2$. For Markov hopping patterns, p_h is simply $2/q$ [12].

Now, consider a receiver perfectly synchronized with the first transmitter. Our goal is to compute the average probability that a binary symbol sent on a particular hop of duration T by the first transmitter will be received in error by this receiver given that BFSK modulation with noncoherent detection is used.

The channel model as seen by the first user is shown in Fig. 1. Also, the noncoherent demodulator which is optimal for AWGN channels is shown in Fig. 2 [13]. The complex low-pass equivalent of the received signal for the jth hop interval $[jT, (j+1)T)$ by the first receiver may be written as follows,

$$r(t) = \sum_{k=1}^{K'+1} \alpha_k p(t - jT - \tau_k) \exp(i 2\pi b_k \Delta f (t - \tau_k) + i\varphi_k) + z(t) \qquad (1)$$

where $\alpha_k > 0, b_k \in \{-1, +1\}, \tau_k \in (-T, T)$, and $\varphi_k \in [0, 2\pi)$ are, respectively, the amplitude, data bit, the delay and the initial phase of the kth transmitter, $i = \sqrt{-1}$ and $p(t) = 1$ for $t \in [0, T]$ and zero, otherwise. The frequency separation between the two FSK signals is denoted by $2\Delta f$. The background noise is incorporated as $z(t)$ which is the equivalent low-pass Gaussian noise process with power spectral density N_0. Since the receiver is assumed to be perfectly synchronized with the first transmitter, $\tau_1 = 0$.

The outputs of the two correlators corresponding to $b_1 = 1$ and $b_1 = -1$ are denoted by U_1 and U_{-1}, and tedious algebra shows that given a $b_1 = +1$ was transmitted U_1, U_{-1}

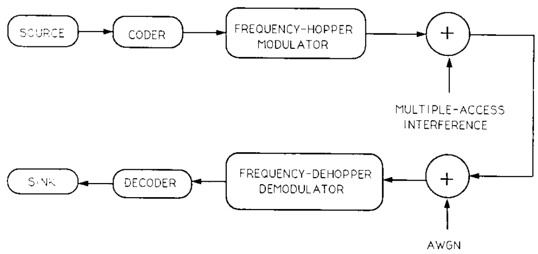


Fig. 1. The channel model as seen by the first user.

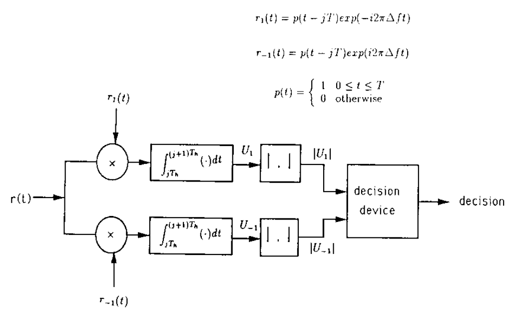


Fig. 2. Noncoherent demodulator.

can be written as

$$U_1 = z_1 + e^{i\varphi_1} + \sum_{k=2}^{K'+1} \left(\frac{\alpha_k}{\alpha_1}\right) \exp(i(\theta(1, p_k, b_k) + \varphi_k)) \cdot A(1, p_k, b_k) \quad (2)$$

$$U_{-1} = z_{-1} + \rho e^{i\varphi_1} + \sum_{k=2}^{K'+1} \left(\frac{\alpha_k}{\alpha_1}\right) \exp(i(\theta(-1, p_k, b_k) + \varphi_k)) \cdot A(-1, p_k, b_k) \quad (3)$$

where $\zeta = 2\Delta fT$ is the normalized separation between the two BFSK signals and $E_b = \alpha_1^2 T/2$. The term ρ is the complex correlation coefficient between the two BFSK signals defined as $\rho = \exp(-i\pi\zeta)\mathrm{sinc}(\zeta)$ where $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. Also, z_1 and z_{-1} are zero mean complex Gaussian random variables with $E\{z_1\bar{z}_1\} = E\{z_{-1}\bar{z}_{-1}\} = 1/(E_b/N_0), E\{z_1\bar{z}_{-1}\} = \rho$. The normalized delay of the kth signal is denoted by $p_k = \tau_k/T$. The magnitude and phase contributions of the kth interfering user, $A(l, p_k, b_k)$ and $\theta(l, p_k, b_k)$, $l \in \{-1, 1\}$, to the correlator outputs is given as follows:

$$A(l, p_k, b_k) = \begin{cases} q_k \,\mathrm{sinc}(\zeta q_k) & b_k = l \\ q_k & b_k \neq l \end{cases} \quad (4)$$

$$\theta(l, p_k, b_k) = \begin{cases} -l\pi\zeta & b_k \neq l \\ -l\pi p_k \zeta & b_k = l \end{cases} \quad (5)$$

where $q_k = 1 - |p_k|$. We first compute the conditional probability of error conditioned on $\boldsymbol{p} = \{p_2, p_3, \cdots, p_{K'+1}\}, \boldsymbol{b} = \{b_2, b_3, \cdots, b_{K'+1}\}$ and $\boldsymbol{K}$. When $K' = 1$ and $\zeta = 1$, the decision variables $|U_1|$ and $|U_{-1}|$ are independent (see Appendix). We assume that they are independent for $K' > 1$ and $\zeta < 1$. Under this assumption we can rewrite $|U_1|$ and $|U_{-1}|$ as

$$U_1 = z_1 + e^{i\varphi_1^{(1)}} + \sum_{k=2}^{K'+1} \left(\frac{\alpha_k}{\alpha_1}\right) \exp\left(i\varphi_k^{(1)}\right) A(1, p_k, b_k) \quad (6)$$

$$U_{-1} = z_{-1} + |\rho| e^{i\varphi_1^{(-1)}} + \sum_{k=2}^{K'+1} \left(\frac{\alpha_k}{\alpha_1}\right) \exp\left(i\varphi_k^{(-1)}\right) A(-1, p_k, b_k) \quad (7)$$

where $\{\varphi_k^{(l)}\}$ are independent and identically distributed (i.i.d.) with a uniform distribution on $[0, 2\pi)$. Now it is clear that U_1 and U_{-1} are sums of independent and spherically symmetric complex random variables and hence they themselves are spherically symmetric conditioned on $\boldsymbol{p}, \boldsymbol{b}$ and $\boldsymbol{K}$ [10], [11].

Since the noncoherent detector chooses $\mathrm{argmax}_{l=-1,1}\{|U_l|\}$ as its estimate, the probability of error conditioned on $\boldsymbol{p}, \boldsymbol{b}$ and $\boldsymbol{K}$ can be written as follows:

$$\begin{aligned} P_e(\boldsymbol{p}, \boldsymbol{b}, \boldsymbol{K}) = & \Pr\{|U_{-1}| > |U_1| | b_1 = +1, \boldsymbol{p}, \boldsymbol{b}, \boldsymbol{K}\} \\ & \cdot \Pr\{b_1 = +1\} \\ & + \Pr\{|U_1| > |U_{-1}| | b_1 = -1, \boldsymbol{p}, \boldsymbol{b}, \boldsymbol{K}\} \\ & \cdot \Pr\{b_1 = -1\}. \end{aligned} \quad (8)$$

Assuming that the data bits of the first user are equiprobable, $P_e(\boldsymbol{p}, \boldsymbol{b}, \boldsymbol{K})$ can be simplified to

$$P_e(\boldsymbol{p}, \boldsymbol{b}, \boldsymbol{K}) = \Pr\{|U_{-1}| > |U_1| | b_1 = +1, \boldsymbol{p}, \boldsymbol{b}, \boldsymbol{K}\}. \quad (9)$$

Hence, the probability of error can be approximated as folows (for $K' \geq 2$) by assuming that $|U_1|$ and $|U_{-1}|$ are independent [7], [10]

$$P_e(\boldsymbol{p}, \boldsymbol{b}, \boldsymbol{K}) \simeq -\int_0^\infty \Phi_1(s) \frac{d\Phi_{-1}(s)}{ds}\, ds \quad (10)$$

where $\Phi_1(s)$ and $\Phi_{-1}(s)$ are the characteristic functions of U_1 and U_{-1}, respectively. Let us define $\overline{\Phi}_1(s)$ and $\overline{\Phi}_{-1}(s)$ to be

$$\overline{\Phi}_1(s) = \Pi_{k=2}^{K'} J_0(A(1, p_k, b_k)s) \quad (11)$$

$$\overline{\Phi}_{-1}(s) = \Pi_{k=2}^{K'} J_0(A(-1, p_k, b_k)s) \quad (12)$$

which are the characteristic functions of the contributions of the multiple-access interference on the decision variables, and $J_n(\cdot)$ denotes the Bessel function of the first kind of order n defined by [14]

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(z\sin(\theta) - n\theta)\, d\theta. \quad (13)$$

Then $\Phi_1(s)$ and $\Phi_{-1}(s)$ can be easily shown by

$$\Phi_1(s) = e^{-s^2/4\gamma} J_0(s) \overline{\Phi}_1(s) \quad (14)$$

$$\Phi_{-1}(s) = e^{-s^2/4\gamma} J_0(|\rho| s) \overline{\Phi}_{-1}(s) \quad (15)$$

where $\gamma = E_b/N_0$. Hence, $d\Phi_{-1}(s)/ds$ can be computed to be

$$\frac{d\Phi_{-1}(s)}{ds} = -e^{-s^2/4\gamma}\left[\left(\frac{s}{2\gamma}J_0(|\rho|s) + |\rho|J_1(|\rho|s)\right)\overline{\Phi}_{-1}(s) - J_0(|\rho|s)\frac{d\overline{\Phi}_{-1}(s)}{ds}\right]. \quad (16)$$

Now if we further define $\overline{\Phi}_{1,-1}(s)$ as

$$\overline{\Phi}_{1,-1}(s) = \overline{\Phi}_1(s)\left[\left(\frac{s}{2\gamma}J_0(|\rho|s) + |\rho|J_1(|\rho|s)\right)\overline{\Phi}_{-1}(s) - J_0(|\rho|s)\frac{\overline{\Phi}_{-1}(s)}{ds}\right], \quad (17)$$

then $P_e(\boldsymbol{p}, \boldsymbol{b}, \boldsymbol{K})$ can be written as

$$P_e(\boldsymbol{p}, \boldsymbol{b}, \boldsymbol{K}) = \int_0^\infty e^{-s^2/2\gamma}J_0(s)\overline{\Phi}_{1,-1}(s)ds. \quad (18)$$

Straightforward computation yields

$$\frac{d\overline{\Phi}_{-1}(s)}{ds} = -\sum_{k=2}^{K'} A(-1, p_k, b_k)J_1(A(-1, p_k, b_k)s) \cdot \Pi_{l\neq k}J_0(A(-1, p_l, b_l)s). \quad (19)$$

Hence $\overline{\Phi}_{1,-1}(s)$ is given by

$$\overline{\Phi}_{1,-1}(s) = T_1(s) \cdot \Pi_{k=2}^{K'+1}T_2(p_k, b_k, s) + J_0(|\rho|s) \cdot \sum_{k=2}^{K'+1} [T_3(p_k, b_k, s) \cdot \Pi_{l\neq k}T_2(p_l, b_l, s)] \quad (20)$$

where

$$T_1(s) = \left[\frac{s}{2\gamma}J_0(|\rho|s) + |\rho|J_1(|\rho|s)\right] \quad (21)$$

$$T_2(p_k, b_k, s) = J_0(A(1, p_k, b_k)s)J_0(A(-1, p_k, b_k)s) \quad (22)$$

$$T_3(p_k, b_k, s) = A(-1, p_k, b_k)J_1(A(-1, p_k, b_k)s) \cdot J_0(A(1, p_k, b_k)s). \quad (23)$$

This completes the derivation of the probability of error conditioned on $\boldsymbol{p}, \boldsymbol{b}, \boldsymbol{K}$ (18).

Now, in order to average (18) over $\boldsymbol{p}$ and $\boldsymbol{b}$, we need to evaluate $P_e(\boldsymbol{K}) = E_{\boldsymbol{p},\boldsymbol{b}}\{P_e(\boldsymbol{p}, \boldsymbol{b}, \boldsymbol{K})\}$ which can be written as

$$E_{\boldsymbol{p},\boldsymbol{b}}\{P_e(\boldsymbol{p}, \boldsymbol{b}, \boldsymbol{K})\} = E_{\boldsymbol{p},\boldsymbol{b}}\left\{\int_0^\infty e^{-s^2/2\gamma}J_0(s)\overline{\Phi}_{1,-1}(s)\right\}ds \quad (24)$$

$$= \int_0^\infty e^{-s^2/2\gamma}J_0(s)E_{\boldsymbol{p},\boldsymbol{b}}\{\overline{\Phi}_{1,-1}(s)\}\, ds \quad (25)$$

where $E_{\boldsymbol{p},\boldsymbol{b}}$ denotes the expectation with respect to $\boldsymbol{p}$ and $\boldsymbol{b}$. Using our assumption that p_k's are i.i.d. and uniform on $(-1,1)$ and b_k's are i.i.d. and equally likely to be -1 or $+1$, and also that $\boldsymbol{p}$ and $\boldsymbol{b}$ are independent of each other and to all other random variables, we can write $E_{\boldsymbol{p},\boldsymbol{b}}\{\overline{\Phi}_{1,-1}(s)\}$ as follows:

$$E_{\boldsymbol{p},\boldsymbol{b}}\{\overline{\Phi}_{1,-1}(s)\} = T_1(s) \cdot \Pi_{k=2}^{K'+1}E_2(k, s) + J_0(|\rho|s) \cdot \sum_{k=2}^{K'+1} [E_3(k, s) \cdot \Pi_{l\neq k}E_2(l, s)] \quad (26)$$

where

$$E_2(k, s) = E_{\boldsymbol{p},\boldsymbol{b}}\{T_2(p, b, s)\} \quad (27)$$

$$E_3(k, s) = E_{\boldsymbol{p},\boldsymbol{b}}\{T_3(p, b, s)\} \quad (28)$$

and we dropped the dependence of p_k's and b_k's on k in the notation. Now if we group the product and the summation into groups of equal power levels, we have

$$E_{\boldsymbol{p},\boldsymbol{b}}\{\overline{\Phi}_{1,-1}(s)\} = T_1(s) \cdot \Pi_{l=1}^{N}E_2(l, s) + J_0(|\rho|s)\sum_{l=1}^{N}\left[\overline{k}_l E_3(l, s) \cdot E_2(l, s)^{\overline{k}_l - 1} \cdot \Pi_{j\neq l}E_2(j, s)^{\overline{k}_j}\right] \quad (29)$$

where the parameters l and j in E_2 and E_3 now denote the power level group. Hence, for each point s we choose in our numerical integration of (25), we need to compute $E_2(l, s)$ and $E_3(l, s)$ for each power level of l. Note that these two terms are independent of K' and the signal-to-noise ratio.

It is worth while at this point to see how (29) would be simplified if all the users in the network had the same power. If we set $\alpha_k = \alpha_1$ for all $k = 1, 2, \cdots K$, then $E_{\boldsymbol{p},\boldsymbol{b}}\{\overline{\Phi}_{1,-1}(s)\}$ simplifies to

$$E_{\boldsymbol{p},\boldsymbol{b}}\{\overline{\Phi}_{1,-1}(s)\} = T_1(s) \cdot E_2(s)^{K'} + J_0(|\rho|s) \cdot K' \cdot E_3(s) \cdot E_2(s)^{K'-1} \quad (30)$$

where we further dropped the dependence of E_2, E_3 on the power levels. We may further simplify this expression for the case when orthogonal BFSK is employed, i.e., $\rho = 0$, as

$$E_{\boldsymbol{p},\boldsymbol{b}}\{\overline{\Phi}_{1,-1}(s)\} = \left(\frac{s}{2\gamma}\right)E_2(s)^{K'} + K' \cdot E_3(s) \cdot E_2(s)^{K'-1}. \quad (31)$$

In this case we may compute $P_e(\boldsymbol{K}) = P_e(K')$, the probability of error for a hop given that the hop is hit by K' users with the same power as the first user using (25) and (31) and thus compute the average probability of error P_e for the first user using

$$P_e = \sum_{K'=0}^{K-1} \binom{K-1}{K'} p_h^{K'}(1-p_h)^{(K-K')}P_e(K'). \quad (32)$$

Going back to the general case where the power levels are different, we need to further average (25) over $\boldsymbol{K}$ and thus the average probability of error can be written as[2] [7]

$$P_e = E_{\boldsymbol{K}}\{P_e(\boldsymbol{K})\} \tag{33}$$

$$= \sum_{\overline{k}_1=0}^{\overline{K}_1} \cdots \sum_{\overline{k}_N=0}^{\overline{K}_N} P_e(\boldsymbol{K}) \Pi_{l=1}^{N} \left\{ \binom{\overline{K}_l}{\overline{k}_l} p_h^{\overline{k}_l} \cdot (1-p_h)^{\overline{K}_l - \overline{k}_l} \right\} \tag{34}$$

$$= \int_0^\infty e^{-s^2/2\gamma} J_0(s) \times \left[\sum_{\overline{k}_1=0}^{\overline{K}_1} \cdots \sum_{\overline{k}_N=0}^{\overline{K}_N} E_{p,b}\{\overline{\Phi}_{1,-1}(s)\} \Pi_{l=1}^{N} \cdot \left\{ \binom{\overline{K}_l}{\overline{k}_l} p_h^{\overline{k}_l} (1-p_h)^{\overline{K}_l - \overline{k}_l} \right\} \right] ds. \tag{35}$$

Now let us discuss the reason for considering nonorthogonal FSK where $|\rho| > 0$. The basic idea behind using nonorthogonal FSK in AFHSS–MA is to provide a tradeoff between the number of errors caused by background noise and multiple-access hits. If we assume that the error probability is $1/2$ whenever a hop is hit, the fact that the system performance will be dominated by multiple-access interference rather than background noise for sufficiently large signal-to-noise ratios leads to the following idea. If we use nonorthogonal FSK instead of orthogonal FSK, we will be able to increase the number of slots available for a given fixed bandwidth. This in effect reduces p_h. This is done at the cost of higher probability of error since increasing $|\rho|$ increases the probability of error when the hop is not hit [13] (it also increases probability of error when the hop is hit). These competing factors will result in an optimum $|\rho|$ that minimizes the average error probability and since the multiple-access hits are far more detrimental than the quiescent errors under the 1/2-bound, the optimal $|\rho|$ is expected to be greater than zero under this assumption for a reasonably high signal-to-noise ratio. In general this optimum $|\rho|$ will depend on the number of users in the network and the signal-to-noise ratio. Since $\zeta = 2\Delta f T$, we may write

$$\zeta = \frac{\Delta f}{\Delta f_{\text{ortho}}} \tag{36}$$

where $\Delta f \le \Delta f_{\text{ortho}}$ is the frequency separation employed and $2f_{\text{ortho}} = 1/T$ is the minimum frequency separation needed for the signals to be orthogonal. Then the approximate number of slots available is given by

$$q_n = \left\lfloor \frac{q_{\text{ortho}}}{\frac{1}{2}(1+\zeta)} \right\rfloor \ge q_{\text{ortho}} \tag{37}$$

where q_{ortho} is the number of slots available when $\Delta f = \Delta f_{\text{ortho}}$ and $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Thus, with Markov hopping patterns, p_h decreases from $2/q_{\text{ortho}}$ to $2/q$ as we decrease the frequency separation from Δf_{ortho} to Δf. Another way of making $|\rho| > 0$ is to fix $\Delta f = \Delta f_{\text{ortho}}$ and decrease the hop duration from T_{ortho} to T where T_{ortho} is the minimum hop duration required to guarantee that the two BFSK signals are orthogonal. This idea of using nonorthogonal signaling is interesting. Since it can be shown, using the 1/2-bound, that the gain achieved by making $|\rho| > 0$ is high when coding is employed and background noise is small. However when a more accurate approximation is used it is observed in Section III that we may actually degrade performance by using nonorthogonal BFSK. Significant gains may be obtainable for systems employing other modulation schemes such as slow frequency hopping where the 1/2-bound is thought to be more realistic.

[2] Loose upper and lower bound on P_e can be obtained by setting $P_e(K)$ equal to 1 or 0 for $K' \ge 2$.

A. Memoryless Hopping Patterns

Up to now, we have considered Markov hopping pattterns to simplify the analyses and to obtain results that are readily computed. We may follow similar steps as before and derive expressions for the probability of error when memoryless hopping patterns are employed [15]. But here, let us consider an upper bound and a lower bound on the error probability for the memoryless hopping pattern case. If we can show that the upper bound and the lower bound are tight and also that the error probability when Markov hopping patterns are employed falls in between these bounds, then we will have in effect shown that the probability of error when Markov hopping patterns are employed is a good approximation to the probability of error when memoryless hopping patterns are employed. For the case where orthogonal BFSK is employed and all the users in the network have the same power as seen by the first receiver, a simple bound can be derived where we assume that whenever a hop is overlapped by an interfering user for its entire duration, the error probability is 1 or 0 resulting in an upper bound and a lower bound, respectively. Then it is easy to see that the average error probability will be upper bounded as

$$P_e \le \sum_{K'=0}^{K-1} \binom{K-1}{K'} p_p^{K'} (1-p_p-p_f)^{K-1-K'} P_e(K') + \left[1-(1-p_f)^{K-1}\right] \tag{38}$$

and lower bounded as

$$P_e \ge \sum_{K'=0}^{K-1} \binom{K-1}{K'} p_p^{K'} (1-p_p-p_f)^{K-1-K'} P_e(K') \tag{39}$$

where p_p and p_f are the probability of partial and full hits given by [16]

$$p_p = \frac{2}{q}\left(1-\frac{1}{q}\right) \tag{40}$$

$$p_f = \frac{1}{q^2} \tag{41}$$

and $P_e(K')$ is the probability of error given that the hop is hit by K' partial hits which has already been computed. The numerical results given in the next section show that for all

K, the probability of error when Markov hopping patterns are assumed is indeed a good approximation to the probability of error when memoryless hopping patterns are employed.

B. Synchronous Hopping

An FHSS-MA network where the hopping instances of all the users are synchronized is a special case of the AFHSS-MA network considered above. The error performance of such a system may be obtained by setting all p_k's equal to zero in the expressions for the error probability derived for AFHSS-MA networks and setting $p_h = 1/q$. It is not hard to see that U_1 and U_{-1} are statistically independent in this case and the expressions for the probability of error derived here are exact for all K' when $\zeta = 1$.

C. Asymptotic Performance

It is also possible to consider the asymptotic average probability of error as $K, q \to \infty$ but with $K/q = \lambda$ fixed as in [2] for the case when orthogonal BFSK is used and all users have the same power level as seen by the first user. Let us first define $F(s)$ as follows:

$$F(s) = e^{-s^2/2\gamma} J_0(s)\left[\frac{s}{2\gamma}J_0(|\rho|s) + |\rho|J_1(|\rho|s)\right]. \quad (42)$$

Then we can show that the average error probability may be written as

$$P_e(K) = \sum_{K'=0}^{K-1}\binom{K-1}{K'}p_h^{K'}(1-p_h)^{(K-K')} \times \int_0^\infty \left[F(s)E_2(s)^{K'} + J_0(|\rho|s)\cdot K' \cdot E_3(s)E_1^{K'-1}(s)\right]ds. \quad (43)$$

Straightforward analysis shows that

$$\lim_{K,q\to\infty,\frac{K}{q}=\lambda} P_e(K) = \int_0^\infty e^{-2\lambda(1-E_2(s))} \cdot [F(s) + 2\lambda J_0(|\rho|s)E_3(s)]\,ds. \quad (44)$$

Unfortunately, evaluating this integral is rather difficult but computations using sample values for K, q, and λ show that the probability of error as a function of λ remains essentially unchanged for $q \geq 100$.

III. Numerical Results

First let us consider orthogonal BFSK, that is, the case when $\rho = 0$. We use (18) to compute the probability of error for a given hop as a function of the normalized delay p when there is one interfering user of the same power level with the data bit of the interfering user as the parameter [16]. For signal-to-noise ratio equal to 11 dB, this is shown in Fig. 3 for $p > 0$ along with simulation results where 10 000 errors were collected for each data point. The error probability is symmetric about $p = 0$ since it is a function of $|p|$. In this case, (18) provides exact results and the simulation is performed only to confirm the accuracy of the numerical computation. We note that the error probability is highly dependent on the data bit and the delay of the interfering user.

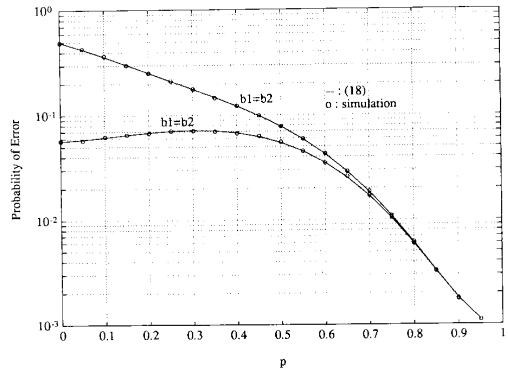


Fig. 3. Probability of error versus the normalized delay p with one interfering user. $E_b/N_0 = 11$ dB.

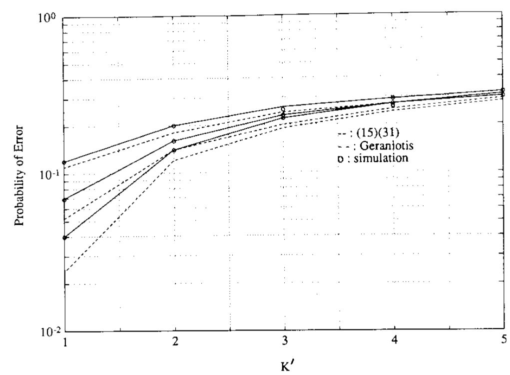


Fig. 4. Probability of error versus the number of interfering users for $E_b/N_0 = 8$, 13, and 19 dB. All users have same power and $\rho = 0$.

TABLE I
COMPARISON OF GERANIOTIS' APPROXIMATION AND (25), (31). $E_b/N_0 = 11$ dB

K'	Geraniotis'	(25), (31)	Simulation
1	6.88×10^{-2}	8.59×10^{-2}	8.50×10^{-2}
2	0.15	0.17	0.17
3	0.21	0.24	0.24
4	0.26	0.28	0.28
5	0.29	0.31	0.31

Fig. 4. shows the plots for the probability of error computed using (25) and (31) and Geraniotis' approximation which is obtained by setting

$$A(1, p_k, -1) = A(-1, p_k, 1) = 0 \quad (45)$$

for $K' = 1, \cdots, 5$ and $E_b/N_0 = 8, 13,$ and 19 dB. Simulation results are also plotted where 20 000 errors were collected for each data point. Again orthogonal BFSK with $2\Delta f = 1/T$ and equal power levels are assumed in these plots.

TABLE II
AVERAGE PROBABILITY OF ERROR FOR TWO POWER LEVEL GROUPS. $E_b/N_0 = 11$ dB; $q = 100$

$\frac{\alpha_2}{\alpha_1}$	K	$\overline{K}_1, \overline{K}_2$	P_e*(35)	*Assuming equal power	*1/2-bound
		(5,5)	1.04×10^{-2}		
	11	(3,7)	7.39×10^{-2}	1.79×10^{-2}	9.22×10^{-2}
		(7,3)	1.34×10^{-2}		
0.5		(10,10)	2.03×10^{-2}		
	21	(5,15)	1.27×10^{-2}	3.49×10^{-2}	0.167
		(15,5)	2.76×10^{-2}		
		(5,5)	2.93×10^{-2}		
	11	(3,7)	3.37×10^{-2}	1.79×10^{-2}	9.22×10^{-2}
		(7,3)	2.47×10^{-2}		
1.5		(10 10)	5.57×10^{-2}		
	21	(5,15)	6.58×10^{-2}	3.49×10^{-2}	0.167
		(15,5)	4.54×10^{-2}		

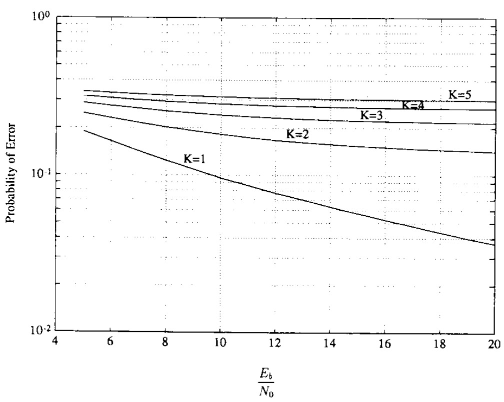


Fig. 5. Probability of error versus E_b/N_0 with K' as the parameter. All users have same power and $\rho = 0$.

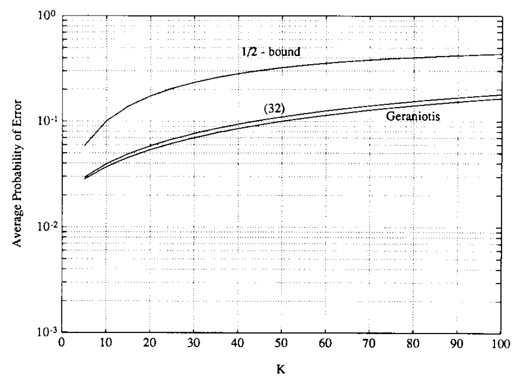


Fig. 6. Average probability of error given that there are K active users in the network with the same power level and $\rho = 0$. $E_b/N_0 = 8$ dB. $q = 100$.

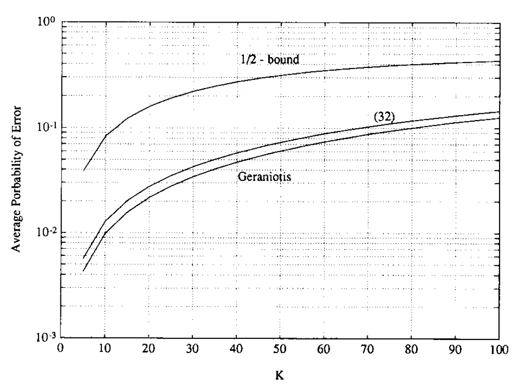


Fig. 7. Average probability of error given that there are K active users in the network with the same power level and $\rho = 0$. $E_b/N_0 = 13$ dB. $q = 100$.

We observe that the values obtained using the approximations (for $K' \geq 2$) derived in this paper fit nicely to the simulation results and Geraniotis' approximation gives optimistic results. These probabilities for $E_b/N_0 = 11$ dB are also tabulated in Table I. Note that the real error probabilities are much smaller than 1/2 for small K'. Since we have checked that the probability of error computed using the expressions developed in this paper fits the simulation results very closely for the signal-to-noise ratios considered, from here on, we will only consider our results using Geraniotis' assumption and compare them to the results obtained using the 1/2-bound. The probability of error as a function of E_b/N_0 is also plotted in Fig. 5.

Let $q = 100$ and consider the average probability of error. In Figs. 6–8 we show the average probability of error of a hop given that there are K users in the network for $E_b/N_0 = 8$, 13, and 19 dB computed using (32). We note that there could be up to an order of magnitude difference between the actual values and the ones obtained using the 1/2-bound for high signal-to-noise ratios. With error correction coding, the difference will be further amplified.

Next, we consider the case where there are two different power level groups P_1 and P_2. Let $\alpha^{(1)}$ and $\alpha^{(2)}$ be the magnitudes of the signals in each group as seen by the first receiver. We take P_1 to have the same power as the first user, i.e., $a^{(1)} = a_1$. We consider two cases where $\alpha^{(2)}/a_1 = 0.5$, $\alpha^{(2)}/a_1 = 1.5$. We tabulate the results for the average error probabilities for these cases for $K = 11$ and

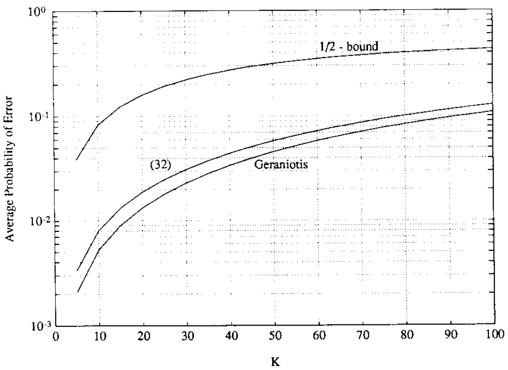


Fig. 8. Average probability of error given that there are K active users in the network with the same power level and $\rho = 0$. $E_b/N_0 = 19$ dB. $q = 100$.

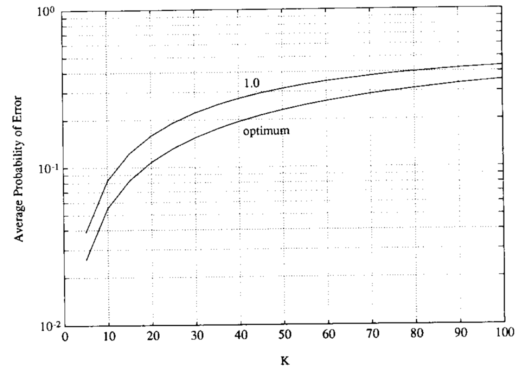


Fig. 10. Average probability of error using nonorthogonal BFSK using the 1/2-bound. All users have equal power. $E_b/N_0 = 19$ dB. $q = 100$.

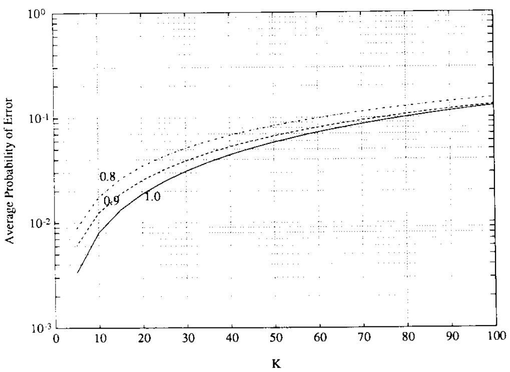


Fig. 9. Average probability of error using nonorthogonal BFSK. All users have equal power. $E_b/N_0 = 19$ dB. $q = 100$. $\mu = 1, 0.9, 0.8$.

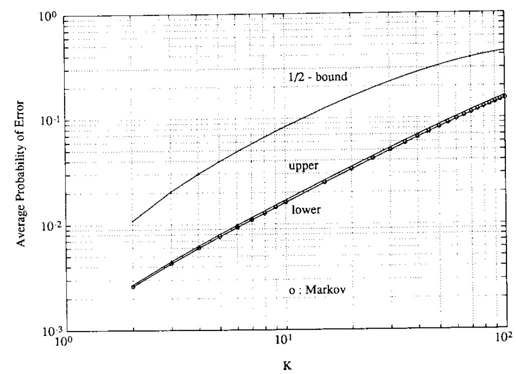


Fig. 11. Bounds on the average probability of error using independent hopping patterns. All users have same power and $\rho = 0$. $E_b/N_0 = 11$ dB. $q = 100$.

$K = 21$ for various combinations of $\overline{K}_1, \overline{K}_2$ in Table II when orthogonal BFSK is employed. The signal-to-noise ratio is taken to be 11 dB. Also included in the table are the results using the 1/2-bound, and the results when equal power among the users is assumed. We note that the errors resulting from these approximations are quite large and again they will further be amplified with the use of error correction coding.

In Fig. 9, the average error probability is shown for the cases when $\mu = 1, 0.9$, and 0.8. We note that employing nonorthogonal BFSK actually degrades performance. Fig. 10 shows similar curves when using the 1/2-bound for the error probability when hit. For the bottom curve of Fig. 10, μ was optimized for each K for minimum average error probability. This shows that conclusions we obtain using the 1/2-bound could be quite different from what we obtain by using our approximation to the error probability when $|\rho| > 0$. In this case, the difference arises from the fact that when the expressions for the probability of error developed in this paper are used, the decrease in the hit probability by using $\mu < 1$ does not sufficiently compensate for the increase in the error probability (both when the hop is hit and when the hop is not hit). More significant gains may be expected for systems where the symbol error probability is close to $(M-1)/M$ whenever a hop is hit such as a slow frequency-hopping system.

In Fig. 11, the bounds on the error probability when memoryless hopping patterns are employed given by (38), (39) are shown. These results verify that the probability of error when Markov hopping patterns are employed is indeed a very good approximation to the probability of error when memoryless hopping patterns are employed for sufficiently large q. The signal-to-noise ratio was taken to be 11 dB.

Most of the computing time needed to evaluate the average error probability is consumed in evaluating (27) and (28) which are independent of K' and the signal-to-noise ratio when Markov hopping patterns are employed. Hence, once these values are computed for each s and the power levels, it is a simple task to obtain the average error probability for different number of interfering users and signal-to-noise ratios. Hence the computing time increases approximately linearly with the number of different power levels and is almost irrelevant to the number of interfering users and the signal-to-noise ratio.

A. Channel Capacity and Associated Throughput

Now we consider the coded performance of AFHSS-MA networks by considering the channel capacity and the normalized throughput as performance measures. When the detector simply makes hard decisions on each hop, then the resulting channel can be accurately modeled as a memoryless binary symmetric channel (BSC) shown in Fig. 12 (independence assumption). When the detector makes erasures by erasing those hops that were hit and makes hard decisions on those hops that were not hit, the resulting channel can be modeled as a memoryless binary symmetric errors and erasure channel (BSEEC) shown in Fig. 13. The channel capacity is the maximum code rate at which there exists a channel code (the best possible code) that achieves reliable communications over the channel. The channel capacity C_{BSC} and C_{BSEEC} for the BSC and the BSEEC are given by [17].

$$C_{\text{BSC}} = 1 + (1-p)\log_2(1-p) + p\log_2 p, \tag{46}$$

$$C_{\text{BSEEC}} = p_c \log_2\left(\frac{2p_c}{1-p_x}\right) + p_e \log_2\left(\frac{2p_e}{1-p_x}\right). \tag{47}$$

We define the normalized throughput of the channel associated with this best possible channel code as the average number of successfully transmitted data bits per channel symbol per frequency slot over the network using this code which is given by [3]

$$w_c = \frac{C \cdot K}{q} \tag{48}$$

where C is the capacity of the channel and K is the number of active users in the network.

In Figs. 14–15, we show the channel capacity and the associated normalized throughput for a system making hard decisions on each hop (system I) and a system using perfect side-information to erase those hops that were hit and making hard decisions on the symbols that were not hit (system II) for $E_b/N_0 = 8$ and 10 dB. It is assumed that all users have the same power as seen by the receiver and orthogonal BFSK is used. We note from these figures that, surprisingly, system I performs much better than system II which is more complicated to implement. An intuitive explanation for this is that the average probability of error when a hop is hit is not as high as $1/2$ and system II will in effect make excessive erasures. Again, this observation may not hold for slow frequency-hop systems.

In Fig. 16, the normalized throughput associated with channel capacity for FHSS–MA networks with asynchronous and synchronous hopping are shown. Orthogonal BFSK and equal power levels among users are assumed. We note that even though p_h for the synchronous system is only half that of the asynchronous system, its performance is inferior to that of the asynchronous system. This is because even though the hops for the synchronous system are less likely to be hit than the asynchronous system, the probability of error once a hop is hit is much large for the synchronous system. Again, this observation may not hold for slow frequency-hop systems.

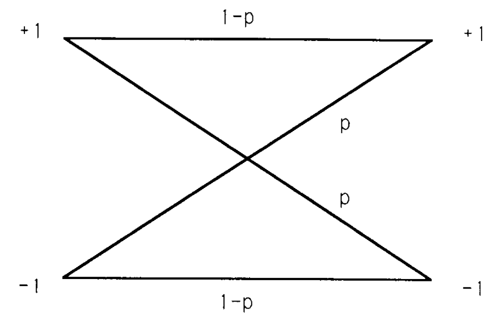


Fig. 12. BSC.

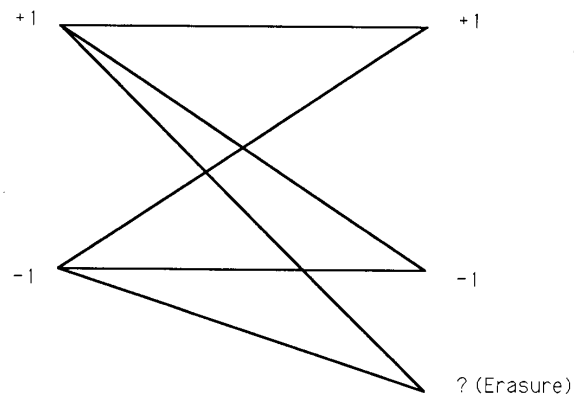


Fig. 13. BSEEC.

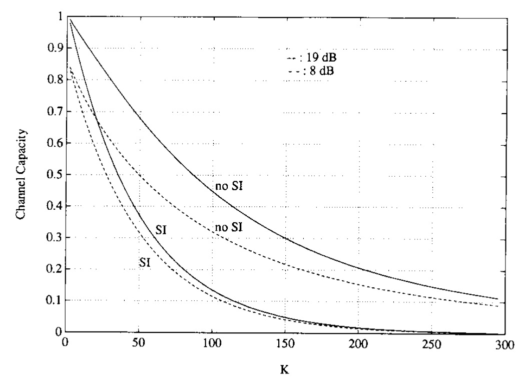


Fig. 14. Channel capacity for $E_b/N_0 = 8$, 19 dB, $q = 100$.

IV. Conclusion

In this paper, we derived accurate approximations for the error probability of an AFHSS–MA network employing Markov hopping patterns transmitting one BFSK modulated symbol per hop. We showed that the bound on the error probability of $1/2$ whenever a hop is hit by multiple-access interference is quite pessimistic for a single bit per hop system. We conjecture that for a system employed M-ary FSK signaling and transmitting one symbol per hop, the usual bound on the error probability of $(M-1)/M$ is also very loose.

The performance of other detection schemes such as

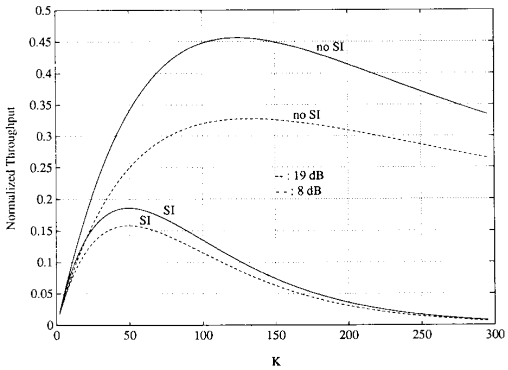


Fig. 15. Normalized throughput associated with channel capacity for $E_b/N_0 = 8, 19$ dB, $q = 100$

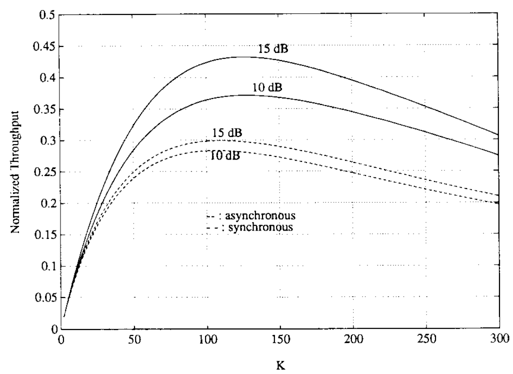


Fig. 16. Normalized throughput associated with channel capacity for synchronous and asynchronous systems for $E_b/N_0 = 10$ dB, $q = 100$.

the Viterbi ratio thresholding in obtaining imperfect side-information in AFHSS-MA networks can be analyzed using the results given in this paper [18].

Appendix

Here we show that $|U_1|$ and $|U_{-1}|$ given by (2), (3) are statistically independent given p_2, b_2 when $\rho = 0$ $K' = 1$. In this case U_1 and U_{-1} are given by

$$U_1 = z_1 + e^{i\varphi_1} + \left(\frac{\alpha_2}{\alpha_1}\right) \exp(i(\theta(1, p_2, b_2) + \varphi_2)) A(1, p_2, b_2) \tag{49}$$

$$U_{-1} = z_{-1} + \left(\frac{\alpha_2}{\alpha_1}\right) \exp(i(\theta(-1, p_2, b_2) + \varphi_2)) A(-1, p_2, b_2) \tag{50}$$

where $\theta(l, p_2, b_2), A(l, p_2, b_2), l \in \{-1, 1\}$ are constants. Let φ be a random variable independent of all the phase and noise terms in (49), (50) with a uniform distribution on $[0, 2\pi)$. Then $U_{-1}e^{i\varphi}$ is given by

$$U_{-1}e^{i\varphi} = z_{-1}e^{i\varphi} + \left(\frac{\alpha_2}{\alpha_1}\right) \exp(i(\theta(-1, p_2, b_2) + \varphi_2 + \varphi)) \cdot A(-1, p_2, b_2). \tag{51}$$

Since we consider all the phase terms $\mathrm{mod}(2\pi)$, $(\varphi_2 + \varphi)$ $\mathrm{mod}(2\pi)$ can be replaced with a random variable φ' which is uniformly distributed on $[0, 2\pi)$ and independent of both φ_2 and φ. This establishes that U_1 and $U_{-1}e^{i\varphi}$ are independent. Hence, since $|U_{-1}| = |U_{-1}e^{i\varphi}|$, $|U_1|$ and $|U_{-1}|$ are independent.

References

[1] E. A. Geraniotis, "Coded FH/SS communications in the presence of combined partial-band noise jamming, Rician nonselective fading, and multiuser interference," *IEEE J. Select. Areas Commun.*, vol. SAC-5, pp. 194–214, Feb. 1987.

[2] S. Kim and W. Stark, "Optimum rate Reed-Solomon codes for frequency-hop spread-spectrum multiple-access communication system," *IEEE Trans. Commun.*, vol. 37, pp. 138–144, Feb. 1989.

[3] M. B. Pursley, "Frequency-hop transmission for satellite packet switching and terrestrial packet radio networks," *IEEE Trans. Inform. Theory*, vol. IT-32, pp. 652–667, Sept. 1986.

[4] M. Hegde and W. E. Stark, "Capacity of frequency-hop spread-spectrum multiple-access communication systems," *IEEE Trans. Commun.*, vol. 38, pp. 1050–1059, July 1990.

[5] E. Geraniotis, "Noncoherent hybrid DS/SFH spread-spectrum multiple-access communications," *IEEE Trans. Commun.*, vol. COM-34, pp. 862-872, Sept. 1986.

[6] C. M. Keller, "An exact analysis of hits in frequency-hopped spread-spectrum multiple-access communications," in *Proc. Conf. Inform. Sci. Syst.*, Mar. 1988, pp. 981–986.

[7] E. Geraniotis, "Multiple-access capability of frequency-hopped spread-spectrum revisited: an analysis of the effect of unequal power levels," *IEEE Trans. Commun.*, vol. 38, pp. 1066–1077, July 1990.

[8] R. Agusti, "On the performance analysis of asynchronous FH-SSMA communications," *IEEE Trans. Commun.*, vol. 37, pp. 488–499, May 1989.

[9] R. T. Short and C. K. Rushforth, "Probability of error for noncoherent frequency-hop spread-spectrum multiple-access communications," Unisys Tech. Rep., PX18829, May 1988.

[10] J. S. Bird, "Error performance of binary NCFSK in the presence of multiple tone interference and system noise," *IEEE Trans. Commun.*, vol. COM-33, pp. 203–209, Mar. 1985.

[11] R. D. Lord, "The use of Hankel transforms in statistics," *Biometrika*, vol. 41, pp. 44–45, 1954.

[12] E. A. Geraniotis and M. B. Pursley, "Error probabilities for slow-frequency-hop spread-spectrum multiple-access communications over fading channels," *IEEE Trans. Commun.*, vol. COM-30, pp. 996–1009, May 1982.

[13] J. G. Proakis, *Digital Communications.* New York: McGraw-Hill, 1983.

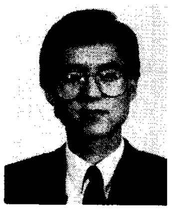
[14] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions.* New York: Dover, 1970 and 1972.

[15] K. Cheun, "Analysis of asynchronous frequency-hop spread-spectrum multiple-access networks," Ph.D. dissertation, EECS Dep. Univ. Michigan, Ann Arbor, MI 48109; also Tech. Rep. Commun. Signal Processing Lab., no. 268, EECS Dep. Univ. Michigan.

[16] E. A. Geraniotis, "Coherent hybrid DS-SFH spread-spectrum multiple-access communications," *IEEE J. Select. Areas Commun.*, vol. SAC-3, pp. 695–705, Sept. 1985.

[17] R. G. Gallager, *Information Theory and Reliable Communications.* New York: Wiley, 1968.

[18] K. Cheun and W. Stark,"Performance of convolutional codes with Viterbi ratio thresholding in asynchronous FHSS-MA," *Proc. 1990 Conf. Inform. Sci. Syst.*, Mar. 1990, pp. 86–91.



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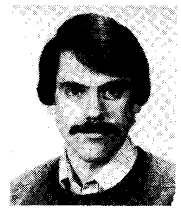

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