On the Error Probability of Coded Frequency-Hopped Spread-Spectrum Multiple-Access Systems

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Abstract—**In this paper, we present a simple exact calculation of the probability distribution of the number of hits in a block of n symbols in a frequency-hopped spread-spectrum multiple-access communication system. While the sequence of hits is not Markovian there is an underlying Markovian structure that allows the probability distribution of the number of hits to be calculated in a recursive fashion. Knowing the probability distribution of the number of hits allows us to calculate the probability of error with error correcting codes for several different types of receivers including receivers with both errors and erasures. The numerical results show that both the approximation obtained by assuming the actual sequence of hits is Markovian and the approximation obtained by assuming the hits are independent are very good. When the number of frequency slots is not too small (<5) our calculations show that the assumption of independence of hits gives an error probability accurate to within 1% of the actual error probability and the assumption that the hits are Markovian gives error probabilities which are accurate to within 0.001%.**

I. Introduction

IN this paper, we consider an asynchronous frequency-hopped spread-spectrum multiple-access communication system employing error correcting codes. The frequency-hopper changes frequency for every code symbol transmitted. It is known that if the number of users K is greater than 2, the hits within a codeword for a particular user are dependent (see [1] and [2]). As shown in [1] the sequence of hits for a particular user alone is not a Markov chain. However, as we have shown in [3], an underlying Markov chain can be identified. This allows, as we will show, for a simple recursive calculation of the error probability. We do this for a general errors and erasures channel with two states. State 1 corresponds to a symbol being hit. State 0 corresponds to no hit. For each state there is an erasure and error probability. Two special cases of interest are the perfect side information case and the no side information case. Perfect side information refers to the case where a hit causes an erasure with probability one with the transmission being noise-free when there is no hit. We say there is no side information when a hit causes an error for a symbol with probability one (and is undetected). In this paper we present numerical results for these two cases. The numerical results show that the approximation obtained by assuming independent symbol errors/erasures is indeed very accurate. The remainder of the paper is organized as follows. First, (in Section II) we set up the model and the notation. In Section III, we derive the recursive formula for the error probability and for the probability distribution of the number of hits in a block of length n. In Section IV we give numerical results and conclusions.

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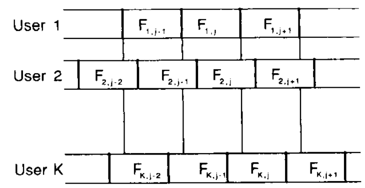

Fig. 1. Timing diagram for frequency hopped multiple access.

II. Model

We first introduce some notation with the aid of Fig. 1. Each user employs a hopping pattern with frequencies chosen uniformly from the set $\{1, \cdots q\}$ and independently of the frequencies chosen by the other users. We denote the random hopping pattern for user 1 as $\{F_{1,j};\ j = \cdots, -2, -1, 0, 1, \cdots\}$. (All capital letters will denote random quantities (variables or vectors), and the corresponding lower case letters will denote particular realizations of these random quantities). Observe that 2 channel symbols of user i overlap with the jth channel symbol of user 1. We define the frequency on the right (see Fig. 1) as $F_{i,j}$ and the collection $(F_{2,j}, \cdots F_{K,j})$ as $\tilde{F}_j$. Now suppose all K users are transmitting packets and a receiver who wants to receive user 1's message locks onto user 1's hopping pattern. We assume that user one's packet consists of the symbols transmitted using frequencies $F_{1,1}, F_{1,2}, \cdots, F_{1,n}$. We denote by H_j, $j = 1, \cdots, n$ binary random variables such that $H_j = 1$ if the jth channel symbol transmitted by user 1 is hit (i.e., if at least one of the other $K - 1$ users uses the same frequency during the slot corresponding to the jth channel symbol) and $H_j = 0$, otherwise. We note that $H_j = 1$ if and only if $F_{1,j} \in \tilde{F}_{j-1} \cup \tilde{F}_j$. We also will need the binary random variables, H_j^L, H_j^R, $j = 1, \cdots, n$ defined as follows:

$$H_j^L = \begin{cases} 1, & F_{1,j} \in \tilde{F}_{j-1} \\ 0, & \text{otherwise} \end{cases}$$

$$H_j^R = \begin{cases} 1, & F_{1,j} \in \tilde{F}_j \\ 0, & \text{otherwise} \end{cases}.$$

Note that $H_j = 1$ iff $H_j^L = 1$ or $H_j^R = 1$. Finally, let S_i^j be the number of frequency slots out of time slots $i, i+1, \cdots, j$ of user 1 that have been hit.

III. Error Probability

We now develop a recursive algorithm for the exact calculation of $P(S_1^n = l)$. Our algorithm utilizes the following key Lemma whose proof may be found in [3].

Lemma 1: (H_j^L, H_j^R) is a Markov chain.

Furthermore, from [3] we know that the stationary Markov chain $U_j = (H_j^L, H_j^R)$ with 4 states, $a = (0, 0)$, $b = (0, 1)$, $c = (1, 0)$, and $d = (1, 1)$, has the following stationary distribution and transi-

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tion probabilities:

$$p_a = \alpha^2$$

$$p_b = \alpha(1-\alpha)$$

$$p_c = \alpha(1-\alpha)$$

$$p_d = (1-\alpha)^2$$

$$p_{a,a} = p_{a,c} = \frac{1}{q}\alpha + \left(1-\frac{1}{q}\right)\beta$$

$$p_{b,a} = p_{b,c} = \left[\frac{1}{q}\alpha + \left(1-\frac{1}{q}\right)\beta\right]\left(\frac{1-\alpha}{\alpha}\right)$$

$$p_{c,a} = p_{c,c} = \left(1-\frac{1}{q}\right)(\alpha-\beta)$$

$$p_{d,a} = p_{d,c} = \left(1-\frac{1}{q}\right)(\alpha-\beta)\left(\frac{1-\alpha}{\alpha}\right)$$

$$p_{a,b} = p_{a,d} = \left(1-\frac{1}{q}\right)(\alpha-\beta)\left(\frac{\alpha}{1-\alpha}\right)$$

$$p_{b,b} = p_{b,d} = \left(1-\frac{1}{q}\right)(\alpha-\beta)$$

$$p_{c,b} = p_{c,d} = \left[\frac{1}{q}(1-\alpha) + \left(1-\frac{1}{q}\right)(2(1-\alpha)-(1-\beta))\right]\left(\frac{\alpha}{1-\alpha}\right)$$

$$p_{d,b} = p_{d,d} = \left[\frac{1}{q}(1-\alpha) + \left(1-\frac{1}{q}\right)(2(1-\alpha)-(1-\beta))\right]$$

where $\alpha = (1-1/q)^{K-1}$, $\beta = (1-2/q)^{K-1}$, p_g is the (stationary) probability, $P(U_j = g)$, and $p_{g,w}$ is the (stationary) transition probability $P(U_{j+1} = g | U_j = w)$. Let S be the set of states, i.e., $S = \{a, b, c, d\}$ and for $0 \le k \le n$, let $r_{j,i}^m(k)$ = probability of starting in state i at time 1 and ending in state j at time m, with state a appearing exactly k times in this finite state sequence of length m. Clearly, from the Markovian nature of U_j, we can write

$$r_{j,i}^{m+1}(k) = \sum_{s \in S} r_{s,i}^m(k) p_{j,s}, \quad j \ne a$$

and

$$r_{j,i}^{m+1}(k) = \sum_{s \in S} r_{s,i}^m(k-1) p_{j,s}, \quad j = a.$$

Thus we can recursively calculate $r_{j,i}^m$ starting with

$$r_{j,i}^2(0) = 1 \quad j \ne a,\ i \ne a$$

$$r_{j,i}^2(1) = 1 \quad j = a,\ i \ne a$$

$$r_{j,i}^2(1) = 1 \quad j \ne a,\ i = a$$

$$r_{j,i}^2(2) = 1 \quad j = a,\ i = a.$$

Finally, we calculate

$$P(S_1^n = n - l) = \sum_{i \in S} \sum_{j \in S} r_{j,i}^n(l) p_i.$$

To calculate the error probability of the decoder we need to know the channel model (i.e., transition probability matrix) from input to output when there is a hit and when there is no hit. As an example consider a M-ary to $M+1$-ary channel transition matrix which makes errors and erasures. If the model is such that the error and erasure probability is $p_{1,e}$ and $p_{1,x}$ respectively, when there is a hit and $p_{0,e}$ and $p_{0,x}$, respectively, when there is no hit then the probability of not decoding correctly P_e is the probability that the number of erasures e plus twice the number of errors t exceeds the error correcting capability of the code, i.e.,

TABLE I
(32, 16) Reed–Solomon with Erasure Correction
$(e = 16)$, $q = 50$

K	P_i	P_m	P_e	
3	0.24442670	0.24657235	0.24644967	$(\times 10^{-10})$
4	0.97537923	0.98549783	0.98548722	$(\times 10^{-8})$
5	0.52829432	0.53371606	0.53371092	$(\times 10^{-6})$
6	0.96046795	0.96963327	0.96962402	$(\times 10^{-5})$
7	0.87746584	0.88499567	0.88498777	$(\times 10^{-4})$
8	0.49965934	0.50342493	0.50342088	$(\times 10^{-3})$
9	0.20171501	0.20302328	0.20302185	$(\times 10^{-2})$
10	0.62744450	0.63087804	0.63087424	$(\times 10^{-2})$

$$\begin{aligned} P_e &= P(e + 2t > d_{\min}) \\ &= \sum_{e,t:\, e+2t > d\min} P(e \text{ erasures, } t \text{ errors}) \\ &= \sum_{\substack{e,t \\ e+2t>d\min}} \sum_{l=0}^{n} P(e \text{ erasures, } t \text{ errors} | S_1^n = l) P(S_1^n = l). \end{aligned}$$

Since $P(S_1^n = l)$ is known from above, what remains is to determine $P(e \text{ erasures, } t \text{ errors} | S_1^n = l)$. This can be calculated as

$$P(e \text{ erasures, } t \text{ errors} | S_1^n = l) = \sum_{e_1=0}^{e} \sum_{t_1=0}^{t} p_{1,l}(e_1, t_1) p_{0,l}(e - e_1, t - t_1)$$

where $p_{1,l}(e, t)$ is the probability of e erasures and t errors in l symbols which have been hit and $p_{0,l}(e, t)$ is the probability of e erasures and t errors in $n - l$ symbols which have not been hit. It is easy to show that

$$p_{1,l}(e, t) = \binom{l}{e,\, t,\, l-t-e} p_{1,e}^e p_{1,x}^t (1 - p_{1,x} - p_{1,x})^{l-e-t}$$

and

$$p_{0,l}(e, t) = \binom{n-l}{e,\, t,\, l-t-e} p_{0,e}^e p_{0,x}^t (1 - p_{0,e} - p_{0,x})^{n-l-e-t}.$$

Two important special cases are when there is an erasure whenever a symbol is hit and when there is an error whenever a symbol is hit (and for both no errors when a symbol is not hit). In both cases the above reduce to

$$P_c = 1 - P_e = \sum_{l=0}^{e} P(S_1^n = l)$$

where $e = n - k$ and k is the dimension of the code for the erasures only case and $e = \lfloor (n-k)/2 \rfloor$ for the errors only case.

IV. Numerical Results and Conclusions

We have evaluated the above expressions for several different sets of system parameters. The results are shown in Tables I-VI. We also show the results obtained in [2] where the error probability was calculated with a Markovian model for the errors (denoted by P_m). (The errors are not Markovian as shown in [1]). Finally, we show the error probability calculated with an independence model for the

TABLE II
(16, 4) REED–SOLOMON WITH ERASURE CORRECTION
$(e = 12)$, $q = 10$

K	P_i	P_m	P_e	
3	0.16676303	0.17316047	0.17312504	$(X\,10^{-3})$
4	0.53471473	0.55329223	0.55315545	$(X\,10^{-2})$
5	0.39636754	0.40615062	0.40607102	$(X\,10^{-1})$
6	0.13622131	0.13826011	0.13824311	$(X\,10^{-0})$
7	0.29663027	0.29881415	0.29879693	$(X\,10^{-0})$
8	0.48439873	0.48544782	0.48544266	$(X\,10^{-0})$
9	0.65675525	0.65617314	0.65618420	$(X\,10^{-0})$
10	0.78955987	0.78775193	0.78777426	$(X\,10^{-0})$

TABLE III
(15, 7) REED–SOLOMON WITH ERASURE CORRECTION
$(e = 8)$, $q = 50$

K	P_i	P_m	P_e	
3	0.33220444	0.33391111	0.33391000	$(X\,10^{-6})$
4	0.86165755	0.86700765	0.86700305	$(X\,10^{-5})$
5	0.77640492	0.78129566	0.78129093	$(X\,10^{-4})$
6	0.39248287	0.39484831	0.39484588	$(X\,10^{-3})$
7	0.13780793	0.13857943	0.13857860	$(X\,10^{-2})$
8	0.37665388	0.37857557	0.37857347	$(X\,10^{-2})$
9	0.85780619	0.86173630	0.86173195	$(X\,10^{-2})$
10	0.17009936	0.17079016	0.17078931	$(X\,10^{-1})$

TABLE IV
(15, 7) REED–SOLOMON WITH ERASURE CORRECTION
$(e = 8)$, $q = 100$

K	P_i	P_m	P_e	
3	0.92313929	0.92570837	0.92569744	$(X\,10^{-9})$
4	0.29130785	0.29234016	0.29233957	$(X\,10^{-7})$
5	0.31867398	0.31988579	0.31988520	$(X\,10^{-6})$
6	0.19515028	0.19590444	0.19590405	$(X\,10^{-5})$
7	0.82817891	0.83135152	0.83134983	$(X\,10^{-5})$
8	0.27293869	0.27396091	0.27396035	$(X\,10^{-4})$
9	0.74767593	0.75038962	0.75038810	$(X\,10^{-4})$
10	0.17787416	0.17849596	0.17849561	$(X\,10^{-3})$

sequence of errors (denoted by P_i). From the table, the following conclusions are evident. Unless one is either working with very small values of q (< 5) or one is interested in knowing the error probability to within less than roughly 0.5% the independence assumption is the model to use to calculate the error probability. However, the recursive method might be an easier way to calculate error probabilities for larger values of n even if independence is valid since it does not need the calculation of binomial coefficients. If one is interested in knowing the error rate to within 0.001% than the Markovian error model will suffice. If one is interested in a more accurate calculation, obviously the exact model should be employed. The Markov model and the exact calculation are essentially within the roundoff error of the computer. As there are many other system parameters which will not be known to within 1% (the accuracy assuming independence) it seems that the independence assumption should be adequate for nearly all purposes.

TABLE V
(31, 15) REED–SOLOMON WITH ERASURE CORRECTION
$(e = 8)$, $q = 50$

K	P_i	P_m	P_e	
3	0.42558507	0.42633287	0.42633265	$(X\,10^{-3})$
4	0.62357070	0.63364012	0.63363971	$(X\,10^{-2})$
5	0.33095979	0.33138681	0.33138664	$(X\,10^{-1})$
6	0.98653594	0.89737437	0.98737406	$(X\,10^{-1})$
7	0.20794990	0.20804580	0.20804578	$(X\,10^{0})$
8	0.34833579	0.34838999	0.34839000	$(X\,10^{0})$
9	0.49777976	0.49774973	0.49774980	$(X\,10^{0})$
10	0.63589453	0.63577182	0.63577195	$(X\,10^{0})$

TABLE VI
(31, 15) REED–SOLOMON WITH ERASURE CORRECTION
$(e = 8)$, $q = 100$

K	P_i	P_m	P_e	
3	0.20930735	0.20954807	0.20954804	$(X\,10^{-5})$
4	0.49725699	0.49792305	0.49792292	$(X\,10^{-4})$
5	0.41057856	0.41111387	0.41111375	$(X\,10^{-3})$
6	0.19030134	0.19052785	0.19052780	$(X\,10^{-2})$
7	0.61308680	0.61372921	0.61372906	$(X\,10^{-2})$
8	0.15388561	0.15402369	0.15402366	$(X\,10^{-1})$
9	0322186346	0.32242764	0.32242758	$(X\,10^{-1})$
10	0.58805320	0.58841087	0.58841079	$(X\,10^{-1})$

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