On Decoding Concatenated Codes

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***Abstract* —The correcting properties of concatenated codes with parallel decoding over an additive channel are investigated. The ith inner decoder's output is a codeword if the Euclidean distance between the received vector and some codeword is less than Δ_i and an erasure otherwise. The outer decoders correct errors and erasures. The error-correcting capability, which is taken to be the minimum length of any noise vector that can cause an error, is obtained for a bank of z inner and outer decoders as a function of the thresholds used. We also find the set of thresholds that maximize the error-correcting capability. We show that for a small number of branches, the error-correcting capability is nearly as large as any decoder.**

I. Introduction

Concatenating codes, which were first investigated by Forney [5], is a way of achieving low error probabilities via long block length (linear) codes at rates below capacity without requiring an impossibly complex decoder. The idea is that the "encoder-channel-decoder" of a normal communication system can be viewed as a kind of superchannel (see Fig. 1) for which a (outer) code is employed. Decoding errors made by the inner decoder can be corrected by the outer decoder. If the length, dimension, and minimum Hamming distance for the inner and outer (block) codes are denoted by $(n_1, k_1, d_{H,1})$ and $(n_2, k_2, d_{H,2})$, respectively, it is then well known that the corresponding concatenated code has length $N = n_1 n_2$, dimension $K = k_1 k_2$, and minimum Hamming distance $D_H \geq d_{H,1} d_{H,2}$ with equality if every nonzero codeword in the inner code has constant weight. The squared Euclidean distance of the concatenated code can be lower bounded in a similar fashion as $D_E^2 \geq d_{H,2} d_{E,1}^2$, where the Euclidean distance of the inner code depends on the Hamming distance of the inner code and the channel on which the code is used. Concatenation is then a technique for producing long codes from short codes.

Decoding a concatenated code is, on first thought, quite easy. One performs the inner decoding and the outer decoding separately. The inner decoder processes the incoming data and uses all the available information to correct random errors and detect burst errors. The output of the inner decoder becomes the input to the outer decoder, which corrects errors and erasures. For a simple M-ary input–output channel, the inner decoder is able to correct $\lfloor (d_{H,1} - 1)/2 \rfloor$ errors. The outer decoder is able to correct $\lfloor (d_{H,2} - 1)/2 \rfloor$ errors, and the combination is then able to correct $\lfloor (d_{H,1} - 1)/2 \rfloor \lfloor (d_{H,2} - 1)/2 \rfloor$ errors, which is roughly $d_{H,1} d_{H,2}/4$ errors. However, the code has minimum distance lower bounded by $d_{H,1} d_{H,2}$, and thus, a minimum Hamming distance decoder should be able to correct $\lfloor (d_{H,1} d_{H,2} - 1)/2 \rfloor$ errors. Similar observations are also true when the inner code is used on an additive channel. This fact led Zyablov to propose a decoding structure that obtained the maximum error-correcting capability for a concatenated code over an M-ary input–output channel. The structure proposed by Zyablov [12] is a collection of z inner–outer decoders operating in parallel. All of the outer decoders are identical errors and erasures-correction decoders. The ith inner decoder's output is an erasure if the received vector is not within Hamming distance Δ_i of some codeword. If the received vector is within distance Δ_i of a codeword, the output of the inner decoder is that codeword. Since Δ_i is less than $d_{H,1}/2$, this codeword will be unique if it exists. Zyablov showed that with appropriate choices for the thresholds, the error-correcting capability of this decoding structure is equal to the lower bound on the minimum Hamming distance of the code. The number of decoders required for this is $z = \lfloor (d_{H,1} + 1)/2 \rfloor$.

Others have taken the same approach as Zyablov for the additive channel except that instead of using minimum Hamming distance for decoding, these approaches used the concept of generalized minimum distance (GMD) proposed by Forney. Furthermore, there have been refinements of the basic algorithm of Forney, which we briefly describe later. We take the same approach as Zyablov for the additive channel but use Euclidean distance as the basic distance instead of Hamming distance or GMD. If a decoder can be found that will correctly decode whenever the received vector is within half the Euclidean distance, then the decoder is asymptotically optimum for binary codes used on the additive white Gaussian noise channel for large signal-to-noise ratios [3], [9], [11].

In our decoder, the ith inner decoder's output is an erasure if the received (inner) vector is not within Euclidean distance Δ_i from some codeword. If the received vector is within Euclidean distance Δ_i of a codeword, then the output of the inner decoder is that codeword. (It will be unique if we require $\Delta_i \leq d_{E,1}/2$.) Our results are, first, a closed-form expression for the minimum Euclidean length of a noise vector that causes an error as a function of both the number of branches in the parallel structure and the thresholds employed. Second, we find a simple algorithm to determine the optimal thresholds and the resulting minimum length noise vector that will cause an error. Third, asymptotic results are derived, and they show that for a large number of branches, the minimum noise vector causing an error approaches half the minimum Euclidean distance of the concatenated code, which is the best one could hope for. Furthermore, our numerical results show that only four decoders operating in parallel are needed to obtain 95% of the optimum error correcting capability. Thus, we essentially obtain a soft decision decoder by only using a soft decision inner decoder (with small block length) and an algebraic (errors and erasure correcting) outer decoder. This type of decoding structure is also very useful for channels with unknown (or varying) statistics [1]–[2], [8]–[10].

The remainder of the paper is organized as follows. In Section II, we begin by describing Forney's GMD decoding algorithm. We also describe other approaches to decoding concatenated codes over an additive channel. All of these alternative approaches are based on Forney's GMD decoding algorithm, which is inherently different from what we consider here (Euclidean distance) since in GMD decoding, one necessarily must limit the input to the decoder to a finite range. Finally, we describe our decoding structure and algorithm. In Section III, we analyze the performance in terms of the minimum Euclidean

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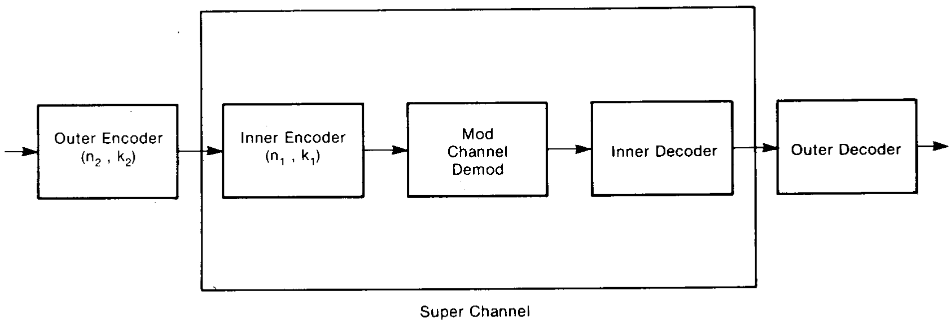

Fig. 1. Concatenated coded system.

length of a noise vector that will cause an error. We refer to this as the error-correcting capability since a noise vector of smaller length will not cause an error. We also find the thresholds for each of the inner decoders that maximizes the error-correcting capability of the decoder and analyze the asymptotic performance. In Section IV, we end with some conclusions and open problems.

II. Decoding Concatenated Codes

In this section, we first describe Zyablov's algorithm mentioned in Section I, which results in maximum error-correcting capability for an M-ary input–output channel. We then describe Forney's GMD decoding algorithm and several modifications thereof. Next, we introduce our notation and describe decoding algorithms based on the GMD for decoding concatenated codes over an additive channel. Finally, we describe our algorithm for decoding concatenated codes over an additive channel.

As is pointed out earlier, Zyablov [13] found a parallel decoding algorithm for M-ary input–output channels, which has the maximum possible error-correcting capability, at the expense of increasing the complexity of the decoder. Zyablov's algorithm depends on errors-and-erasures decoding and the use of several branches with different tentative decisions. The inner decoder of the ith branch corrects all error patterns with $i-1$ or fewer errors for $i=1,2,\cdots,\lfloor (d_{H,1}+1)/2 \rfloor$, and the outer decoder corrects e_i errors and τ_i erasures if $2e_i+\tau_i<d_{H,2}$. The errors-and-erasures-correcting decoder can either produce a codeword or fail. Of the at-most $\lfloor (d_{H,1}+1)/2 \rfloor$ codewords produced, the one that has the smallest Hamming distance from the received vector is taken as the final result. This decoder corrects all error patterns with weight $\lfloor (d_{H,1}d_{H,2}-1)/2 \rfloor$ or less. This is the largest possible error correction possible. A more general treatment of concatenated codes for the M-ary input–output channel can be found in [12].

For a channel that has an infinite output alphabet, such as an additive noise channel, Forney [5] proposed the GMD system, in which the demodulator passes to the decoder its best estimate of the code symbol and a real number, which indicates how reliable it supposes its estimate to be. To describe this algorithm, consider a binary (for simplicity) block code of length n and minimum Hamming distance $d_{\min}$ with code symbols in the alphabet $\{-1,+1\}$. Let $\boldsymbol{y}=(y_1,y_2,\cdots,y_n)$ be the output of the demodulator where $y_i\in\{-1,+1\}$ is an estimate of the ith transmitted symbol. Let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ be a vector with the ith component measuring the reliability of the ith estimate. The reliability is constrained to be in the interval $[0,1]$. Now, let $\bar{y}_i=\alpha_i y_i$ and denote the inner product between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ by $(\boldsymbol{u},\boldsymbol{v})$. The generalized minimum distance between $\boldsymbol{c}$ and $\boldsymbol{y}$, $\boldsymbol{\alpha}$ is defined as (see [7])

$$d_G(\boldsymbol{c},\boldsymbol{y},\boldsymbol{\alpha})=n-(\boldsymbol{c},\bar{\boldsymbol{y}}).$$

Forney shows that for any α satisfying the constraint that $\alpha_i\in[0,1]$ and any $\boldsymbol{y}$, there is, at most, one codeword satisfying

$$d_G(\boldsymbol{c},\boldsymbol{y},\boldsymbol{\alpha})<d_{\min}. \tag{1}$$

The GMD decoder processes the received vectors $\boldsymbol{\alpha},\boldsymbol{y}$ as follows. First, an attempt to decode the vector $\boldsymbol{y}$ is made using an error-correcting decoder. If the output of the decoder $\boldsymbol{c}$ satisfies (1), then $\boldsymbol{c}$ is taken as the transmitted codeword. If the error-correcting decoder fails or the decoded codeword does not satisfy the previous equation, the least reliable symbol is erased, and an attempt to decode with an errors-and-erasures-correcting decoder is made. If the decoded codeword satisfies (1), it is then taken as the transmitted codeword. Otherwise, or if the decoder fails, the two least reliable symbols are erased, and decoding is attempted again. This continues until a codeword is decoded with $d_G(\boldsymbol{c},\boldsymbol{y},\boldsymbol{\alpha})<d_{\min}$ or there are more erasures than can be corrected by the code. Forney shows that if a decoded codeword satisfies the previous inequality, it will be found by this algorithm. As previously described, there is a vector decoded by the inner decoder with i erasures for $i=0,1,2,\cdots,d_{\min}-1$; however, it can be shown that only $\lfloor (d_{\min}+1)/2 \rfloor$ decoding attempts are sufficient to guarantee that the codeword (if it exists) satisfying the above will be found by this technique.

Kovalev [7] proposes two modifications of decoding, with respect to the GMD metric, that incorporate additional complexity constraints. The problem addressed and solved in [7] is the following: For at most $l<\lceil (d_{\min}+1)/2 \rceil$ decoding attempts, what is the decoding algorithm that maximally reduces the possible loss in the realized correcting properties of the code compared with Forney's decoding procedure? In the first case, as in Forney's algorithm, the number of erasures made for each of the decoding attempts is fixed and is independent of the received vector. In the second case, the number of erasures in each decoding attempt depends in some way on the actual received vector. For example, for one received vector, decodings with 1, 3, and 7 erasures could be made, whereas for another received vector, decodings with 1, 3, and 5 erasures could be made. As with GMD decoding, if an output of one of the decoding attempts is such that $d_G<d_{\min}$, that codeword is then the overall output. For the second class of algorithms, Kovalev shows that no loss is incurred compared with Forney's algorithm with fixed thresholds with only $(d_{\min}+1)/4$ decoding attempts and variable thresholds. Our algorithm is similar to the second algorithm of Kovalev in that the number of erasures made in each of the decoding attempts may vary from one received

(concatenated) vector to another. However, it differs in that we do not consider GMD decoding but consider Euclidean distance decoding.

Finally, for a continuous-output channel, an algorithm was proposed by Dumer *et al.* [4] for binary concatenated decoding with respect to the GMD. Both the inner and outer decoders are algebraic decoders that are capable of correcting errors and erasures. Because of the use of GMD decoding, the demodulator output must be limited to a finite range, e.g., $-1 \leq \tilde{y}_i \leq 1$. With a decoder based on Euclidean distance, such limiting need not be done.

We now describe the decoding structure and algorithm analyzed in the next section and introduce some notation. Let z be the number of inner and outer decoders. Consider the parallel decoding structure shown in Fig. 2 such that the inner decoder of the ith branch decodes all vectors within Euclidean distance Δ_i of some codeword $i = 1, 2, \cdots, z$ and detects errors otherwise. We impose the constraint that $\Delta_i \in [0, d_{E,1}/2]$. The parallel decoder is then specified by the set of thresholds $\Lambda_z = \{\Delta_1, \Delta_2, \cdots, \Delta_z\}$, where with no loss of generality, we have $\Delta_0 \triangleq 0 \leq \Delta_1 \leq \cdots \leq \Delta_z \leq \Delta_{z+1} \triangleq d_{E,1} - \Delta_z$. The outer code corrects any set of e errors and τ erasures if $2e + \tau < d_{H,2}$. If $2e + \tau \geq d_{H,2}$, the outer decoder will either produce an incorrect codeword or fail to produce any codeword. Of the at-most z possible decoded (concatenated) codeword results, the one that has the smallest Euclidean distance from the received combination is taken as the final result.

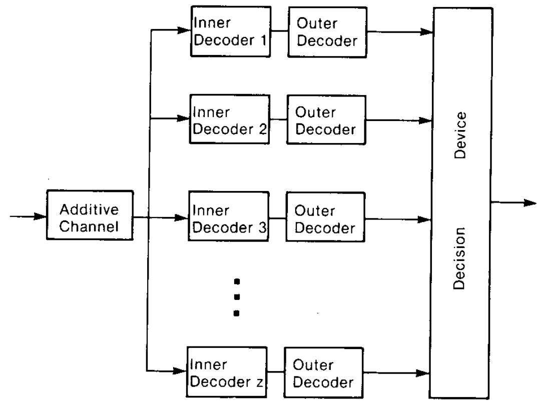

Fig. 2. Parallel concatenated decoding.

III. Error-Correcting Capability

We are interested in evaluating the error-correcting capability of the decoding structure shown in Fig. 2 using the Euclidean distance as a measure of performance, that is, we would like to determine the minimum length of any noise vector that will cause the overall decoded output to be incorrect. The overall decoded output is the outer decoder output that is closest to the received vector in Euclidean distance. This will be incorrect if each of the individual decoders produce incorrect outputs or fail. It is also possible for some of the codewords produced by an outer decoder to be correct, whereas some other (incorrectly) decoded codeword is closer to the received vector. In this case, the length of the noise vector must be greater than half the minimum Euclidean distance of the code. As we will see, the length of the minimum noise vector with the condition that all decoders decode incorrectly will be no greater than half the Euclidean distance. Thus, it is sufficient in determining the minimum length noise vector to assume that all decoders incorrectly decode for the overall decoder to incorrectly decode.

We first find the error-correcting capability for any set of thresholds. We then optimize the thresholds to maximize the correcting capability and then examine the asymptotic performance for a large number of branches.

We begin by finding the error-correcting capability of the concatenated code for an arbitrary set of thresholds. This is done in two steps. First, we find the correcting capability for arbitrary Λ_z, given e_k errors and τ_k erasures in the kth branch of the decoder, $k = 1, \cdots, z$. This is denoted by $\gamma(\Lambda_z, \boldsymbol{\tau}, \boldsymbol{e})$, where $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_z)$ and $\boldsymbol{e} = (e_1, \cdots, e_z)$. Second, we solve for the minimum of $\gamma(\Lambda_z, \boldsymbol{\tau}, \boldsymbol{e})$ over all possible vectors $\boldsymbol{\tau}$ and $\boldsymbol{e}$ satisfying certain constraints. Intuitively, we can reason as follows (see Fig. 3): For a fixed number of errors and erasures, the noise vector of smallest length will be in the direction of the closest (concatenated) codeword. The received concatenated vector consists of n_2 inner received vectors. If there are e_1 received vectors that cause the first decoder to output an incorrect codeword, the received vectors must be at least distance $(d_{E,1} - \Delta_1)$ from the (inner) codeword with equality if the noise vectors in all of these (inner) received vectors are in the direction of the closest (inner) codeword from the transmitted codeword. If there are e_2 received vectors that cause the second inner decoder to produce an incorrect codeword, e_1 of these will

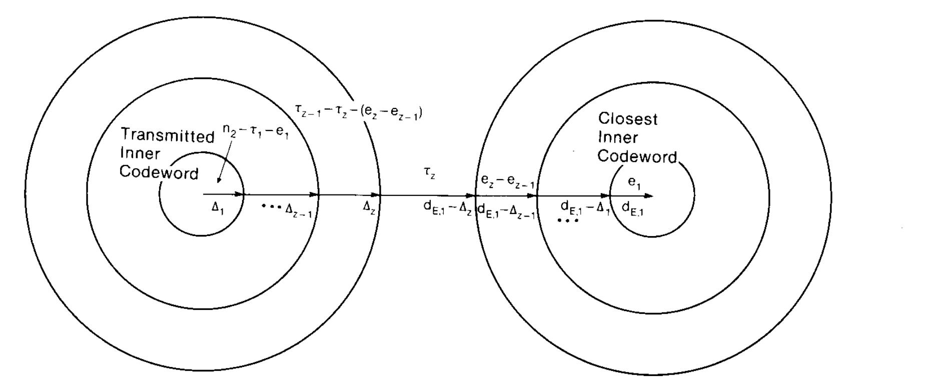

Fig. 3. Decoding regions for inner decodings. τ_i and e_i are number of erasures and errors at output of ith decoder, respectively. Every received vector that causes decoder i to make an error also causes decoder j to make an error for $j > i$. Similarly, every received vector that causes decoder k to make an erasure also causes decoder l to make an erasure for $l < k$.

then also cause the first decoder to produce an incorrect codeword. Thus, $e_2 - e_1$ received vectors will produce erroneous codewords from decoder 2 and will be erased by decoder 1. These received vectors will have to be at least distance $(d_{E,1} - \Delta_2)$ far away from the transmitted codeword with equality if the noise vectors are directed as before. For the zth decoder there will be $e_z - e_{z-1}$ received vectors decoded incorrectly that are erased by the other decoders. These noise vectors must be at least distance $(d_{E,1} - \Delta_z)$ far from the transmitted codeword. There are τ_z received vectors that are erased by the zth decoder. These must be at least Δ_z far away from the transmitted codeword. There are τ_{z-1} vectors that cause decoder $z-1$ to produce an erasure. Of these, τ_z are also erased by the zth decoder, and $e_z - e_{z-1}$ cause an error in the zth decoder but not in decoder $z-1$, and they have already been taken into account. The remaining $\tau_{z-1} - \tau_z - (e_z - e_{z-1})$ must be at least distance Δ_{z-1} far away from the transmitted vectors. Continuing in this line of reasoning, the minimum noise length (squared) needed to cause $\boldsymbol{\tau}$ erasures and $\boldsymbol{e}$ errors can be determined. This is shown rigorously in [6]. Thus, the minimum correcting capability (squared) of the code under the condition that the kth decoding trial has τ_k erasures, e_k errors, and arbitrary thresholds Λ_z is given by

$$\begin{aligned}\gamma(\Lambda_z,\boldsymbol{\tau},\boldsymbol{e}) &= \sum_{k=1}^{z-1} \Delta_k^2(\tau_k - \tau_{k-1} + e_k - e_{k-1}) + \Delta_z^2\tau_z \\ &\quad + \sum_{k=2}^{z}(d_{E,1}-\Delta_k)^2(e_k - e_{k-1}) + (d_{E,1}-\Delta_1)^2 e_1 \\ &= \sum_{k=1}^{z} a(k)\tau_k + b(k)e_k \\ &= \sum_{k=1}^{z}\gamma^{(k)}(\tau_k,e_k),\end{aligned} \tag{2}$$

where $a(k) = \Delta_k^2 - \Delta_{k-1}^2 \geq 0$, and $b(k) = (\Delta_k^2 - \Delta_{k-1}^2 + (d_{E,1} - \Delta_k)^2 - (d_{E,1} - \Delta_{k+1})^2) \geq 0$ since $\Delta_0 = 0$ and $\Delta_{z+1} = d_{E,1} - \Delta_z$. In addition, the following two identities are used later:

$$\sum_{m=1}^{z} b(m) = (d_{E,1} - \Delta_1)^2 \tag{3}$$

$$\sum_{m=l_1}^{l_2}(2a(m) - b(m)) = (d_{E,1} - \Delta_{l_2+1})^2 - (d_{E,1} - \Delta_{l_1})^2 + \Delta_{l_2}^2 - \Delta_{l_1-1}^2. \tag{4}$$

Next, we use this result to determine the correcting capability $\gamma(\Lambda_z,\boldsymbol{\tau})$ under the condition that τ_k erasures occur in the kth branch and decoding is not correct. For all the decoder branches to be incorrect, the following set of z inequalities must hold:

$$e_k \geq \max\left(\frac{d_{H,2} - \tau_k}{2}, 0\right), \qquad k = 1,2,\cdots,z. \tag{5}$$

If we define $\gamma(\Lambda_z,\boldsymbol{\tau})$ as the minimum error correcting capability with τ_i erasures in the ith decoder, then

$$\begin{aligned}\gamma(\Lambda_z,\boldsymbol{\tau}) &= \min_{\boldsymbol{e}}\gamma(\Lambda_z,\boldsymbol{\tau},\boldsymbol{e}) \\ &= \sum_{k=1}^{z}\min_{e_k}\gamma^{(k)}(\tau_k,e_k) \\ &= \sum_{k=1}^{z}\gamma^{(k)}(\tau_k)\end{aligned} \tag{6}$$

where

$$\gamma^{(k)}(\tau_k) = \begin{cases} a(k)\tau_k + b(k)\left\lfloor\dfrac{(d_{H,2} - \tau_k - 1)}{2}\right\rfloor, & 0 \leq \tau_k \leq d_{H,2} \\ a(k)\tau_k, & \tau_k \geq d_{H,2}. \end{cases} \tag{7}$$

The error-correction capability is then found by minimizing $\gamma(\Lambda_z,\boldsymbol{\tau})$ over all vectors $\boldsymbol{\tau}$ such that $\tau_k \geq \tau_{k+1}$. Thus

$$\begin{aligned}\gamma(\Lambda_z) &= \min_{\boldsymbol{\tau}}\gamma(\Lambda_z,\boldsymbol{\tau}) \\ &= \min_{d_{H,2}\geq\tau_1\geq 0}\Big\{\gamma^{(1)}(\tau_1) + \min_{\tau_1\geq\tau_2}\Big\{\gamma^{(2)}(\tau_2) + \cdots \\ &\quad + \min_{\tau_{z-1}\geq\tau_z}\gamma^{(z)}(\tau_z)\Big\}\cdots\Big\}\Big\}.\end{aligned} \tag{8}$$

According to the lemma in Appendix A, the minimum in (8) occurs at point $\boldsymbol{\tau}$, whose elements satisfy

$$\tau_1 \equiv \tau_2 \equiv \cdots \equiv \tau_z \equiv d_{H,2} (\mathrm{mod}\, 2). \tag{9}$$

Let $\Psi = \{\boldsymbol{\tau}: \tau_i \equiv d_{H,2}(\mathrm{mod}\,2), 1 \leq i \leq z\}$. Then

$$\gamma(\Lambda_z) = \sum_{k=1}^{z}\frac{d_{H,2}}{2}b(k) + \min_{\boldsymbol{\tau}\in\Psi}\sum_{m=1}^{z}\frac{\tau_m}{2}(2a(m) - b(m)).$$

Using (3), we have that

$$\gamma(\Lambda_z) = \frac{d_{H,2}(d_{E,1} - \Delta_1)^2}{2} + \min_{\boldsymbol{\tau}\in\Psi}\sum_{m=1}^{z}\frac{\tau_m}{2}(2a(m) - b(m)). \tag{10}$$

Two cases for $d_{H,2}$ are considered.

Case 1: $d_{H,2} \equiv 0(\mathrm{mod}\,2)$. According to lemma, the minimum in this case is attained at one of the $z+1$ points

$$\boldsymbol{\tau} \in S = \{(0,0,\cdots,0),(d_{H,2},0,\cdots,0), (d_{H,2},d_{H,2},0,\cdots,0),\cdots,(d_{H,2},\cdots,d_{H,2})\}.$$

Thus, using (4)

$$\begin{aligned}\gamma(\Lambda_z) &= \frac{d_{H,2}(d_{E,1} - \Delta_1)^2}{2} + \min_{\boldsymbol{\tau}\in S}\sum_{m=1}^{z}\frac{\tau_m}{2}(2a(m) - b(m)) \\ &= \frac{d_{H,2}(d_{E,1} - \Delta_1)^2}{2} + \min_{0\leq k\leq z}\sum_{m=1}^{k}\frac{d_{H,2}}{2}(2a(m) - b(m)) \\ &= \frac{d_{H,2}(d_{E,1} - \Delta_1)^2}{2} + \frac{d_{H,2}}{2} \\ &= \min_{1\leq k\leq z+1}\left((d_{E,1} - \Delta_k)^2 - (d_{E,1} - \Delta_1)^2 + \Delta_{k-1}^2\right) \\ &= \min_{1\leq k\leq z+1}\frac{d_{H,2}}{2}\left((d_{E,1} - \Delta_k)^2 + \Delta_{k-1}^2\right)\end{aligned} \tag{11}$$

where the sum from $m = 1$ to 0, as well as the sum from $m = z+1$ to z used in the next case, are recognized as the empty sum, i.e., 0.

Case 2: $d_{H,2} \equiv 1(\mathrm{mod}\,2)$. In this case

$$\boldsymbol{\tau} \in S' = \{(1,1,\cdots,1),(d_{H,2},1,\cdots,1), (d_{H,2},d_{H,2},1,\cdots,1),\cdots,(d_{H,2},\cdots,d_{H,2})\}$$

and the correcting capability is given by

$$\gamma(\Lambda_z)=\sum_{k=1}^{z}\frac{d_{H,2}}{2}b(k)+\min_{1\le k\le z+1}\left(\sum_{m=1}^{k-1}\frac{d_{H,2}}{2}(2(a(m)-b(m))+\sum_{m=k}^{z}\frac{1}{2}(2a(m)-b(m))\right).$$

The first sum is given in (3), whereas the last two can be determined from (4). Hence,

$$\begin{aligned}\gamma(\Lambda_z)&=\frac{d_{H,2}(d_{E,1}-\Delta_1)^2}{2}\\&\quad+\frac{1}{2}\min_{1\le k\le z+1}\left[d_{H,2}\left((d_{E,1}-\Delta_k)^2-(d_{E,1}-\Delta_1)^2+\Delta_{k-1}^2\right)\right.\\&\quad\left.+(d_{E,1}-\Delta_{z+1})^2-(d_{E,1}-\Delta_k)^2+\Delta_z^2-\Delta_{k-1}^2\right]\\&=\min_{1\le k\le z+1}\frac{(d_{H,2}-1)}{2}\left((d_{E,1}-\Delta_k)^2+\Delta_{k-1}^2\right)+\Delta_z^2.\end{aligned}\tag{12}$$

Thus, we have proven the following theorem.

Theorem 1: For the parallel decoder structure with z branches and fixed thresholds Λ_z, the error-correcting capability is given by

$$\gamma(\Lambda_z)=\begin{cases}\min_{1\le k\le z+1}\dfrac{d_{H,2}}{2}\left((d_{E,1}-\Delta_k)^2+\Delta_{k-1}^2\right),\\ \quad d_{H,2}\text{ even}\\ \min_{1\le k\le z+1}\dfrac{(d_{H,2}-1)}{2}\left((d_{E,1}-\Delta_k)^2+\Delta_{k-1}^2\right)+\Delta_z^2,\\ \quad d_{H,2}\text{ odd.}\end{cases}\tag{13}$$

Our next goal is to find the set of thresholds for a given value of z that maximizes the error-correcting capability. We will obtain analytical solutions for $z=1$ and 2 and state an algorithm that will find the solution for any z.

First, consider the case of $d_{H,2}$ even. The optimal decoder is then the solution of

$$\Lambda_z^*=\arg\max_{\Lambda_z}\gamma(\Lambda_z)$$

with the resulting correcting capability

$$\begin{aligned}\gamma&=\max_{\Lambda_z}\min_{1\le k\le z+1}\frac{d_{H,2}}{2}\left[(d_{E,1}-\Delta_k)^2+\Delta_{k-1}^2\right]\\&=\max_{\Lambda_z}\min_{1\le k\le z+1}\frac{d_{H,2}d_{E,1}^2}{2}\left[(1-\delta_k)^2+\delta_{k-1}^2\right],\end{aligned}\tag{14}$$

where $\delta_k=\Delta_k/d_{E,1}$. Let $f(\delta_k,\delta_{k-1})=(1-\delta_k)^2+\delta_{k-1}^2$. It is shown in Appendix B that

$$\max_{\Lambda_z}\min_{1\le k\le z+1}f(\delta_k,\delta_{k-1})$$

is obtained when

$$f(\delta_k,\delta_{k-1})=\alpha_z,\qquad k=1,2,\cdots,z+1,\tag{15}$$

where α_z is some unique constant that depends only on z. It is not too difficult to show [6] that the optimal thresholds when $d_{H,2}$ is odd also must satisfy (15). Thus, to evaluate the optimum error-correcting capability and the corresponding optimal thresholds, it is sufficient to solve the system of nonlinear equations

$$\begin{aligned}(1-\delta_1)^2&=\alpha\\(1-\delta_2)^2+\delta_1^2&=\alpha\\&\vdots\\(1-\delta_z)^2+\delta_{z-1}^2&=\alpha\\2\delta_z^2&=\alpha.\end{aligned}\tag{16}$$

The error-correcting capability (squared) of the code when the decoder uses its optimal thresholds is

$$\gamma=\frac{d_{H,2}d_{E,1}^2}{2}\alpha_z.$$

We can factor out half the minimum distance of the code to obtain

$$\sqrt{\gamma}=\frac{d_{E,1}\sqrt{d_{H,2}}}{2}\beta_z\tag{17}$$

where $\beta_z=\sqrt{2\alpha_z}\le 1$. We will say a decoder achieves $x\%$ of the full error-correcting capability if $\beta_z=x/100$. The solution to (16) must satisfy the three constraints listed in Appendix B. The desired solution of these equations can be found analytically for $z=1$ to be $\delta_1=\sqrt{2}-1=0.414$, $\beta_1=\sqrt{2\alpha_1}=\sqrt{4\delta_1^2}=0.828$, and for $z=2$, $\alpha_2=0.419$ is the positive solution of $\alpha_2^2-8(2+8\sqrt{2})\alpha_2+16=0$, giving $\beta_2=0.915$. In addition, $\delta_1=1-\sqrt{\alpha_2}=0.353$ and $\delta_2=\sqrt{\alpha_2/2}=0.458$. To describe an algorithm to solve these equations for arbitrary z, consider the following function $g_z(\alpha)$ defined on the interval $[0,1/2]$:

$$\begin{aligned}g_z(\alpha)&=(1-x_1)^2\\x_i&=\sqrt{\max\left\{0,\alpha-(1-x_{i+1})^2\right\}}\qquad i=1,2,\cdots,z-1\\x_z&=\sqrt{\alpha/2}.\\x_z&=\sqrt{\alpha/2}.\end{aligned}$$

Note that $g_z(0)=1$ and $g_z(1/2)=1/4$. In addition, if α_z is the solution of $g_z(\alpha)=\alpha$, then $g_{z+1}(\alpha_z)=1$. Because $g_z(\alpha)-\alpha$ is a continuous nonincreasing function of α, which is positive at 0 and negative at $1/2$, there exists a unique solution α_z to $g_z(\alpha)-\alpha=0$ with $0\le\alpha_z\le 1/2$. This is also the unique solution for α to (16) satisfying the constraints. From this solution, it is easy to find δ_i, $1\le i\le z$ via

$$\begin{aligned}\delta_z&=\sqrt{\alpha_z/2}\\\delta_i&=\sqrt{\alpha_z-(1-x_{i+1})^2}\qquad i=1,2,\cdots,z-1.\end{aligned}$$

Furthermore, these δ_i are the unique solutions to (15) that satisfy the constraints of Appendix B. Thus, a simple algorithm for finding the unique optimal thresholds and the error-correcting capability is the following.

Algorithm:

1) Find the solution α_z to $g_z(\alpha)-\alpha=0$;
2) $\Delta_z=\sqrt{\alpha_z/2}$;
3) $\Delta_i=\sqrt{\alpha_z-(1-\Delta_{i+1})^2}$, $1\le i\le z-1$.

TABLE I
ERROR-CORRECTING CAPABILITY FOR VARIOUS NUMBER OF BRANCHES z

z	α_z	β_z
1	0.3431	0.828
2	0.4187	0.915
3	0.4500	0.948
4	0.4655	0.965
5	0.4750	0.974
6	0.4800	0.979
7	0.4850	0.984
8	0.4877	0.988
9	0.4900	0.990
10	0.4915	0.991
15	0.4957	0.995

With a divide-and-conquer-type algorithm for finding the solution to $g_z(\alpha)-\alpha=0$ in n steps, we can know α_z with accuracy of $n+1$ significant bits. The algorithm could be put into a loop with control variable z. In this case, the interval that needs to be searched to find α_z is the interval $[\alpha_{z-1}, 1/2]$. We have used the algorithm described above to evaluate the error-correcting capability. Table I shows the error-correcting capability as a fraction of the maximum correcting capability. In particular, for $z=3$, we have 94.8% of the maximum with $\delta_1=0.32940$, $\delta_2=0.41600$, and $\delta_3=0.47410$. For $z=4$, we obtain 96.5% of the full error-correcting capability with thresholds $\delta_1=0.317$, $\delta_2=0.395$, $\delta_3=0.443$, and $\delta_4=0.481$.

Finally, we analyze the asymptotic performance achieved by this decoding structure when the number of branches becomes large. The following theorem identifies the asymptotic error-correcting capability:

Theorem 2:

$$\lim_{z\to\infty} \beta_z = 1.$$

Proof: First, we show that α_z is an increasing function of z and is bounded above by $1/2$. That it is bounded above by $1/2$ is shown in Appendix C. That it is increasing is the result of the fact that α_{z+1} is the solution to the equation $g_{z+1}(\alpha)-\alpha=0$, where g_z is a decreasing function, and that $g_{z+1}(\alpha_z)=1$. Thus, the solution $\alpha_{z+1}>\alpha_z$. Because α_z is increasing and bounded, it must converge as $z\to\infty$ to the solution of

$$\alpha_\infty=(1-\delta_\infty)^2+\delta_\infty^2$$
$$\alpha_\infty=2\delta_\infty^2.$$

Solving these two equations for α_∞, we get

$$\alpha_\infty=0.5$$

and the theorem follows. Consequently, the full error-correcting capability of the concatenated code is achieved asymptotically.

IV. CONCLUSION

The error-correcting properties of nonbinary concatenated codes was evaluated for soft decision (inner) decoding. We developed an algorithm that uses errors-and-erasures decoding and the use of several branches with different tentative decisions (i.e., parallel decoding). This decoding algorithm is optimal for a fixed number of decoders.

The algorithm developed by Zyablov for hard decision decoding (based on Hamming distance) is capable of maximum random error correction, which is achieved with a finite number of decoders. Although the maximum error correction (based on Euclidean distance) in our case is only realized asymptotically with the number of decoders, Table I shows that only four decoders are needed to achieve over 95% of the maximum correction capability.

The algorithm proposed in this paper looks at the Euclidean distance correctable by the code as a measure of performance (referred to as the error-correcting capability of the code). Unfortunately, the analysis presented here gives us little insight into how the parallel decoding structure will process received vectors that are not within the minimum Euclidean distance of some codeword. For large signal-to-noise ratios, these regions are not important but are important for moderate signal-to-noise ratios. With some codes, large portions of the space of received vectors are not within half the minimum Euclidean distance of any codeword, and yet, a maximum likelihood decoder would decode these into some codeword as opposed to failing to decode into any codeword. The problem of finding a decoding structure that closely approximates a minimum error probability decoder by utilizing short maximum likelihood (inner) decoders and algebraic-type outer decoders is still quite challenging.

APPENDIX A

In this appendix, we prove a simple lemma.

Lemma: Let $\Upsilon=\{\tau: \tau\equiv d(\text{mod}\,2)\}$, $\Upsilon(x,y)=\{\tau: \tau\in\Upsilon,\ x\le\tau\le y\}$, and $0<a<b$. Then, we have

$$\min_{x\le\tau\le y}\{a\tau+b\lfloor(d-\tau+1)/2\rfloor\}$$
$$=\frac{db}{2}+\min_{\tau\in\Upsilon(x,y)}\tau(2a-b)/2$$
$$=\frac{db}{2}+\frac{1}{2}\min\{x^*(2a-b),y^*(2a-b)\}$$

where $x^*=\min\{\tau\in\Upsilon: \tau\ge x\}$, and $y^*=\max\{\tau\in\Upsilon: \tau\le y\}$.

Proof: Let $f(\tau)\triangleq a\tau+b\lfloor(d-\tau+1)/2\rfloor$. Then, for $\tau^*\in\Upsilon$

$$f(\tau^*-1)=a(\tau^*-1)+b\lfloor(d-\tau^*+1+1)/2\rfloor$$
$$=a\tau^*+b(d-\tau^*)/2+b-a$$
$$f(\tau^*)=a\tau^*+b\lfloor(d-\tau^*+1)/2\rfloor=a\tau^*+b(d-\tau^*)/2$$
$$f(\tau^*+1)=a(\tau^*+1)+b\lfloor(d-\tau^*-1+1)/2\rfloor$$
$$=a\tau^*+b(d-\tau^*)/2+a.$$

The lemma follows from the relations $f(\tau^*-1)>f(\tau^*)$ and $f(\tau^*+1)>f(\tau^*)$.

APPENDIX B

Here, we shall show the optimal decoder thresholds are the solution of a set of nonlinear equations. Define $f(\delta_k,\delta_{k-1})=(1-\delta_k)^2+\delta_{k-1}^2$ for $k=1,2,\cdots,z+1$ with δ_k satisfying the following properties:

1) $\delta_k\in[0,1/2]$ for $k=1,2,\cdots,z$.
2) $\delta_0=0$.
3) $\delta_{z+1}=1-\delta_z$.

In addition, define $\Lambda_z=\{\delta_1,\delta_2,\cdots,\delta_z\}$. We need to find

$$\max_{\Lambda_z}\min_k f(\delta_k,\delta_{k-1}). \tag{18}$$

We will show that it is necessary to have $f(\delta_k,\delta_{k-1})=\alpha$, $k=1,\cdots,z+1$, in which α is some constant. It is shown in Section III that a unique solution satisfying the previous three constraints exists.

Lemma: The solution Λ_z to the optimization $\max_{\Lambda_z} \min_{1 \le k \le z+1} f(\delta_k, \delta_{k-1})$ is the solution to the set of nonlinear equations

$$f(\delta_k, \delta_{k-1}) = \alpha, \qquad k = 1, 2, \cdots, z+1.$$

Proof: Let Λ_z and α (that depends on z) be the unique solution to

$$f(\delta_k, \delta_{k-1}) = \alpha, \qquad k = 1, 2, \cdots, z+1.$$

To prove the lemma, we need to show that for any $\Lambda'_z \ne \Lambda_z$ [1]

$$\min_k f(\delta'_k, \delta'_{k-1}) < \alpha$$

satisfying the previous three constraints. For any set $\Lambda'_z \ne \Lambda_z$, we have $\delta_j > \delta'_j$ or $\delta_j < \delta'_j$, for at least one j. Now let

$$j = \arg\min_k \{\delta_k : \delta_k < \delta'_k\}.$$

If such j exists, then it is obvious that

$$(1-\delta'_j)^2 + \delta'^2_{j-1} < (1-\delta_j)^2 + \delta^2_{j-1} = \alpha.$$

If such j does not exist, let

$$l = \arg\max_k \{\delta_k : \delta_k > \delta'_k\}.$$

Then, we have, $(1-\delta'_{l+1})^2 + \delta'^2_l < (1-\delta_{l+1})^2 + \delta^2_l = \alpha$; we know that l or j, or both, exist. Hence, the optimal set is Λ_z.

Appendix C

Lemma: The solution to the set of (15) is such that $\alpha \le 1/2$.

Proof: First notice that

$$\alpha = \max_{\Lambda_z} \min_k f(\delta_k, \delta_{k-1}) \le \min_k \max_{\Lambda_z} f(\delta_k, \delta_{k-1}).$$

Furthermore

$$\begin{aligned} &\min_k \max_{\Lambda_z} \{(1-\delta_k)^2 + \delta^2_{k-1}\} \\ &\le \min\Big\{ \max_{0 \le \delta_1 \le 1/2} (1-\delta_1)^2, \\ &\quad \max_{0 \le \delta_1 \le \delta_2 \le 1/2} (1-\delta_2)^2 + \delta_1^2, \cdots, \max_{0 \le \delta_{z-1} \le \delta_z \le 1/2}, 2\delta_z^2 \Big\} \\ &= \min\left\{1, 1, \cdots, 1, \frac{1}{2}\right\} \\ &= \frac{1}{2}. \end{aligned}$$

The second equality follows from the fact that

$$\max_{0 \le x \le y \le .5} \{(1-y)^2 + x^2\} = 1.$$

Thus $\alpha \le 1/2$.

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Some Results on the Norm of Codes

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***Abstract*—Two upper bounds for the norm $N(C)$ of a binary linear code C with minimal weight d and covering radius R are given: $N(C) \le 2R + \lceil d/2 \rceil - 1$, $(d \ge 3)$, and $N(C) \le 3R - 2$, $(R \ge 3)$. The second bound implies that C is normal if $R = 3$.**

I. Introduction, An Example

All codes considered in this correspondence are binary and linear. Let C be an $[n,k]$ code with covering radius R (i.e., an $[n,k]R$ code) and minimal weight d. Then the norm of C is defined to be

$$N(C) = \min_{1 \le i \le n} \left[\max_{x \in CF(2)^n} \left(d(x, C^{(i)}) + d(x, C \setminus C^{(i)}) \right) \right] \quad (1.1)$$

where $C^{(i)} = \{(c_1, \cdots, c_n) \in C | c_i = 0\}$. $(d(x, \phi) = n$ by convention.) C is said to be normal if $N(C) \le 2R+1$. The ith coordinate of C is said to be acceptable if $\max_{x \in GF(2)^n} (d(x, C^{(i)}) + d(x, C \setminus C^{(i)})) \le 2R + 1$. The reason for studying normal codes is that the amalgamated direct sum (ADS) construction can be applied to them to generate better covering codes [3]. No example of an abnormal code is known. On the other hand, many codes have been proven to be normal, among which are the codes with $d \le 3$ or $R \le 2$ [2]. Theorem 16 of [5] claimed that if $d = 4$ or 5, all coordinates in the support of a codeword of minimal weight are acceptable. There is a mistake in the proof of this theorem (in the last line on page 624 and the first line on page 625), and we have a counterexample to the conclusion when $d = 5$. The conclusion with $d = 4$ is still true (Corollary 2.3 in this correspondence).

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